DQC1 Normalized Trace Estimation Beyond the Successive-Error Gap

Changpeng Shao

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Abstract

We give two rigorous mechanisms for avoiding the successive-best-approximation condition $E_d/E_{d-1} < 1/2$ in reductions for normalized trace estimation $\mathrm{Tr}[f(A)]/\dim(A)$ of log-local Hamiltonians. The first mechanism uses a degree-D high-accuracy approximant and a uniform Chebyshev clock. It yields the sufficient condition

$$(D - d + 1)(E_D + \eta_{\mathrm{tr}}) \lesssim \varepsilon_0 \delta_U,$$

where d is the ε_0-approximate degree and δ_U is the desired accuracy for the normalized unitary trace. The second mechanism applies to polynomially Lipschitz functions and does not use a higher-degree approximant. An equioscillation dual witness is realized as an endpoint matrix element of an f-dependent open Jacobi matrix. A scale-averaged finite-rank Floquet closure converts that endpoint functional into a constant-strength first Floquet coefficient. We prove a finite-parameter error bound, finite-precision stability, qubit-clock padding, polynomial quadrature, and a positive-weight selector construction that packs the scale and phase averages into a single log-local Hamiltonian. Under an explicit standard uniformity assumption on the construction of the Jacobi witness, inverse-polynomial accuracy normalized trace estimation is DQC1-hard, without any condition on E_d/E_{d-1}.

Logical status

The finite-dimensional mathematical statements in this note are unconditional. A uniform complexity-theoretic hardness statement additionally requires an effective description of the function family and a polynomial-time procedure that outputs the corresponding Jacobi witness to polynomial precision. This uniformity requirement is stated explicitly in Assumption 2.1. No claim is made that an arbitrary Lipschitz function presented only as a black box automatically admits such a construction.

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1 Introduction

For a continuous function $f : [-1,1] \to [-1,1]$, define

$$E_k(f) := \inf_{p \in \mathcal{P}_k} \|f - p\|_\infty, \qquad \widetilde{\deg}_\varepsilon(f) := \min\{k : E_k(f) \le \varepsilon\}.$$

Ji, Li, Shao, Wang, and Zhang [3] prove DQC1-completeness of normalized trace estimation for functions of log-local Hamiltonians under the condition $E_d/E_{d-1} < 1/2$. In their proof, a periodic-Jacobi discriminant creates a signal of size $E_{d-1} - E_d$, while the residual is controlled by E_d.

This note develops two alternatives.

(i) A high-accuracy approximant must possess a non-negligible Chebyshev coefficient in the range from the constant-error approximate degree up to its full degree. A uniform cycle of the matching length filters that coefficient exactly.

(ii) The best-approximation dual witness can instead be encoded in an f-dependent open Jacobi matrix. Closing the chain perturbatively produces f', but averaging over a radial scale integrates the derivative back to f. Lipschitz regularity supplies both inverse-polynomial separation of the alternation nodes and quantitative control of the finite closure parameter.

The second route is the main result. The reduction does not divide by an unknown Floquet coefficient. We orient the endpoint witness so that the coefficient is positive, use the positive phase density $(1 + \cos\phi)/(2\pi)$, compute the U-independent baseline classically from polynomial-size clock matrices, and pack the positive quadrature weights into one selector-controlled Hamiltonian.

2 Preliminaries and conventions

2.1 Approximate degree and the source problem

Fix a constant $\varepsilon_0 \in (0,1)$ and put

$$d := \widetilde{\deg}_{\varepsilon_0}(f).$$

Then

$$E_{d-1}(f) > \varepsilon_0, \qquad E_d(f) \le \varepsilon_0. \tag{1}$$

For an N-dimensional unitary U, write

$$\tau(U) := \frac{\operatorname{Tr} U}{N}.$$

The promise problem of distinguishing $\operatorname{Re}\tau(U) \ge \delta$ from $\operatorname{Re}\tau(U) \le -\delta$, for any inverse-polynomial δ with $0 < \delta < 1$, is DQC1-complete [3, 5].

2.2 Finite-dimensional reduction versus uniform families

The analytic argument will first be stated for one function f and one circuit U. To obtain a uniform complexity theorem for a family f_s, one needs to relate the source size to the index s of the target function. We isolate the necessary computational assumption.

Assumption 2.1 (Effective Jacobi witness)**.** For the function family under consideration, the following tasks can be performed in time polynomial in the family index and the requested number of output bits.

(a) Evaluate $f(x)$ at a rational $x \in [-1,1]$ to the requested precision.

(b) Determine $d = \widetilde{\deg}_{\varepsilon_0}(f)$ and output, to requested operator-norm precision, the Jacobi matrix B constructed from an equioscillation witness in Proposition 4.2.

(c) Output the orientation sign of its endpoint matrix element.

The exact Jacobi matrix exists for every continuous f. The assumption concerns only the uniform bit-complexity of finding it. It can be replaced by any other polynomial-time method that outputs a tridiagonal matrix with the endpoint and gap guarantees used below.

2.3 Qubit encoding of a non-power-of-two clock

A clock of dimension m is encoded in $q = \lceil \log_2 m \rceil$ qubits. Let H act on $\mathbb{C}^m \otimes \mathbb{C}^N$, and extend it by zero on the unused clock basis states:

$$\widehat{H} := H \oplus 0_{(2^q - m)N}.$$

Then

$$\frac{1}{2^q N} \operatorname{Tr} f(\widehat{H}) = \frac{m}{2^q} \frac{1}{mN} \operatorname{Tr} f(H) + \left(1 - \frac{m}{2^q}\right) f(0). \tag{2}$$

The second term is known and independent of every circuit phase, while

$$\frac{1}{2} < \frac{m}{2^q} \leq 1. \tag{3}$$

Thus qubit encoding loses at most a factor two in every nonzero Fourier signal.

3 Uniform Chebyshev clock with a high-accuracy approximant

3.1 A large coefficient in the high-degree tail

Lemma 3.1 (Chebyshev-tail certificate)**.** *Suppose*

$$P_D(x) = \sum_{k=0}^{D} c_k T_k(x), \qquad \|f - P_D\|_\infty \leq E_D, \qquad D \geq d.$$

Then

$$\sum_{k=d}^{D} |c_k| > \varepsilon_0 - E_D. \tag{4}$$

Consequently, some $r \in \{d, \ldots, D\}$ satisfies

$$|c_r| \geq \frac{\varepsilon_0 - E_D}{D - d + 1}. \tag{5}$$

Proof. Let $Q_{d-1} = \sum_{k<d} c_k T_k$. Since $Q_{d-1} \in \mathcal{P}_{d-1}$, $\|f - Q_{d-1}\|_\infty > \varepsilon_0$. Since $\|T_k\|_\infty = 1$,

$$\|f - Q_{d-1}\|_\infty \leq E_D + \sum_{k=d}^{D} |c_k|.$$

This proves (4), and averaging proves (5). □

3.2 The uniform clock is an exact frequency filter

For $r \geq 3$, let

$$J_r(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 1 & & & \mathrm{e}^{-i\theta} \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ \mathrm{e}^{i\theta} & & & 1 & 0 \end{pmatrix}. \tag{6}$$

Its eigenvalues are $\lambda_\ell(\theta) = \cos((\theta + 2\pi\ell)/r)$.

Lemma 3.2 (Exact filtering)**.** *For every $k \geq 0$,*

$$\frac{1}{r} \operatorname{Tr} T_k(J_r(\theta)) = \begin{cases} \cos(q\theta), & k = qr, \\ 0, & r \nmid k. \end{cases} \tag{7}$$

Hence

$$\frac{1}{r} \operatorname{Tr} P_D(J_r(\theta)) = \sum_{q=0}^{\lfloor D/r \rfloor} c_{qr} \cos(q\theta).$$

Proof. Use $T_k(\cos x) = \cos(kx)$ and the geometric sum $\sum_{\ell=0}^{r-1} \mathrm{e}^{2\pi i k\ell/r}$. □

3.3 Reduction and all dimension factors

Let U be an N-dimensional unitary implemented by at most d local gates. Since the selected r satisfies $r \geq d$, the circuit fits in the r-site cycle after padding with identity gates. Let $H_r(U)$ denote the corresponding cyclic history Hamiltonian. If the eigenvalues of U are $\mathrm{e}^{i\theta_j}$, then

$$H_r(U) \simeq \bigoplus_j J_r(\theta_j).$$

For $U_\phi = \mathrm{e}^{i\phi} U$, set

$$Y(\phi) = \frac{1}{rN} \operatorname{Tr} f(H_r(U_\phi)).$$

Writing $Q = \lfloor D/r \rfloor$, one obtains

$$Y(\phi) = c_0 + \sum_{q=1}^{Q} c_{qr} \operatorname{Re}\Big(\mathrm{e}^{iq\phi} \tau(U^q)\Big) + \rho(\phi), \qquad |\rho(\phi)| \leq E_D. \tag{8}$$

For $M = 2Q + 1$ and $\phi_s = 2\pi s/M$, there is no aliasing among frequencies $-Q, \dots, Q$, and

$$\frac{1}{M} \sum_{s=0}^{M-1} \mathrm{e}^{-i\phi_s} Y(\phi_s) = \frac{c_r}{2} \tau(U) + \widehat{\rho}_1, \qquad |\widehat{\rho}_1| \leq E_D. \tag{9}$$

If the clock is stored in $q = \lceil \log_2 r \rceil$ qubits, the left-hand side is multiplied by $r/2^q > 1/2$; the unused-space contribution in (2) is a zero-frequency offset and disappears from (9).

Theorem 3.3 (Uniform-clock sufficient condition)**.** *Assume the coefficient c_r can be computed to sufficient polynomial precision. If each normalized target trace is estimated to error η_{tr}, then $\tau(U)$ is recovered with error*

$$\delta_U \leq \frac{4(E_D + \eta_{\mathrm{tr}})}{|c_r|} \leq \frac{4(D - d + 1)(E_D + \eta_{\mathrm{tr}})}{\varepsilon_0 - E_D}. \tag{10}$$

The factor four includes qubit-clock padding. In particular, if $E_D \leq \varepsilon_0/2$, it suffices that

$$(D - d + 1)(E_D + \eta_{\mathrm{tr}}) \leq \frac{\varepsilon_0 \delta_U}{8}. \tag{11}$$

No ratio between E_d and E_{d-1} is used.

4 The Lipschitz route: exact open Jacobi witness

4.1 Equioscillation and the dual measure

Let $p^* \in \mathcal{P}_{d-1}$ be a best approximant and put $E = E_{d-1}(f) > \varepsilon_0$. By the Chebyshev equioscillation theorem, there are $m = d + 1$ points

$$-1 \le x_1 < x_2 < \cdots < x_m \le 1$$

and a sign such that $f(x_j) - p^*(x_j)$ alternates between E and $-E$. Let

$$\omega(x) = \prod_{j=1}^{m}(x - x_j), \qquad h_j = \frac{\alpha}{\omega'(x_j)}, \qquad \sum_j |h_j| = 1,$$

where the sign of α is chosen so that $h_j(f(x_j) - p^*(x_j)) \ge 0$.

Lemma 4.1 (Discrete dual witness)**.** *For every $q \in \mathcal{P}_{m-2}$,*

$$\sum_{j=1}^{m} h_j q(x_j) = 0, \tag{12}$$

and

$$\sum_{j=1}^{m} h_j f(x_j) = E. \tag{13}$$

Proof. Lagrange interpolation and comparison of the coefficient of x^{m-1} give $\sum_j q(x_j)/\omega'(x_j) = 0$ for $\deg q \le m - 2$. Apply this to p^* and use the alternating signs and $\sum_j |h_j| = 1$. □

Set $w_j = |h_j|$, $D_x = \operatorname{diag}(x_1, \ldots, x_m)$,

$$z = (\sqrt{w_1}, \ldots, \sqrt{w_m})^T, \qquad y = (\operatorname{sgn}(h_1)\sqrt{w_1}, \ldots, \operatorname{sgn}(h_m)\sqrt{w_m})^T.$$

Lanczos orthogonalization of $z, D_x z, \ldots, D_x^{m-1} z$ gives an orthogonal matrix Q and a real symmetric tridiagonal matrix

$$B = Q^T D_x Q. \tag{14}$$

The first Lanczos vector is z. By (12), y is orthogonal to the first $m - 1$ Krylov vectors, so the last Lanczos vector is $\pm y$.

Proposition 4.2 (Open Jacobi endpoint witness)**.** *The matrix B satisfies $\|B\| \le 1$ and*

$$|\langle m|f(B)|1\rangle| = E_{d-1}(f) > \varepsilon_0. \tag{15}$$

Moreover, the sign can be determined from the Lanczos orientation.

Proof. The matrix B is orthogonally similar to D_x. Therefore

$$\langle m|f(B)|1\rangle = \pm y^T f(D_x) z = \pm \sum_j h_j f(x_j),$$

and Lemma 4.1 finishes the proof. □

This is the general-Jacobi form of the endpoint-witness mechanism underlying Theorem 1.12 of Montanaro and Shao [6].

4.2 Lipschitz regularity separates the witness eigenvalues

Assume f is K-Lipschitz. Since $\|f\|_\infty \le 1$ and $E \le 1$, $\|p^*\|_\infty \le 2$. Markov's inequality gives

$$\left\|(p^*)'\right\|_\infty \le 2d^2.$$

At adjacent alternation points,

$$2E \le |f(x_{j+1}) - f(x_j)| + |p^*(x_{j+1}) - p^*(x_j)|.$$

Thus the minimum eigenvalue gap of B satisfies

$$g := \min_j (x_{j+1} - x_j) \ge \frac{2E}{K + 2d^2} > \frac{2\varepsilon_0}{K + 2d^2}. \tag{16}$$

When $K, d = \mathrm{poly}(n)$, both m and $1/g$ are polynomially bounded.

5 Finite-precision stability of the Jacobi witness

A complexity reduction uses rational coefficients. Scalar Lipschitz continuity alone does not imply an operator-Lipschitz estimate for arbitrary noncommuting matrices, so this point must be handled using the simple spectral gap.

Lemma 5.1 (Stability of a spectral matrix function)**.** *Let B and $\widetilde{B}$ be Hermitian $m \times m$ matrices. Suppose B has simple spectrum with minimum gap g, and*

$$\Delta := \left\|\widetilde{B} - B\right\| \le g/4.$$

If $f : [-1,1] \to [-1,1]$ is K-Lipschitz and both spectra lie in $[-1,1]$, then

$$\left\|f(\widetilde{B}) - f(B)\right\| \le K\Delta + \frac{4m\Delta}{g}. \tag{17}$$

The spectrum of $\widetilde{B}$ has minimum gap at least $g/2$.

Proof. Match the eigenvalues λ_j of B and $\widetilde{\lambda}_j$ of $\widetilde{B}$ in increasing order. Weyl's inequality gives $\left|\widetilde{\lambda}_j - \lambda_j\right| \le \Delta$, hence the new gaps are at least $g - 2\Delta \ge g/2$. Let P_j and $\widetilde{P}_j$ be the corresponding rank-one spectral projectors. The Davis–Kahan projector estimate, with the isolated eigenvalue separated by g, gives

$$\left\|\widetilde{P}_j - P_j\right\| \le \frac{4\Delta}{g}.$$

Now write

$$f(\widetilde{B}) - f(B) = \sum_j \big(f(\widetilde{\lambda}_j) - f(\lambda_j)\big)\widetilde{P}_j + \sum_j f(\lambda_j)(\widetilde{P}_j - P_j).$$

The norm of the first sum is at most $K\Delta$ because the projectors $\widetilde{P}_j$ are mutually orthogonal. The triangle inequality bounds the second sum by $4m\Delta/g$. □

Corollary 5.2 (Rational witness)**.** *Let B be the exact matrix in Proposition 4.2. Suppose a rational tridiagonal $\widetilde{B}$ is known with $\left\|\widetilde{B} - B\right\| \le \zeta$. Define*

$$B^\sharp := \frac{\widetilde{B}}{1 + \zeta}.$$

Then $\left\|B^\sharp\right\| \le 1$ and $\left\|B^\sharp - B\right\| \le 2\zeta$. If

$$2\zeta \le \min\left\{\frac{g}{4}, \frac{\varepsilon_0}{4(K + 4m/g)}\right\}, \tag{18}$$

then $B^\sharp$ has gap at least $g/2$ and, after orienting its endpoint sign as for B,

$$\langle m|f(B^\sharp)|1\rangle \geq \frac{3\varepsilon_0}{4}. \tag{19}$$

The required number of coefficient bits is polynomial whenever $m, K, 1/g$ are polynomial and Assumption 2.1 holds.

Proof. The norm and distance statements follow directly from the definition. Apply Lemma 5.1 with $\Delta = 2\zeta$ and use the exact endpoint value greater than ε_0. □

From now on, B denotes either the exact witness or the rationally implemented witness from Corollary 5.2; in the latter case one replaces g by the known lower bound $g/2$ and ε_0 by $3\varepsilon_0/4$. To simplify constants, we retain the notation

$$\|B\| \leq 1, \qquad \operatorname{gap}(B) \geq g, \qquad \langle m|f(B)|1\rangle \geq \varepsilon_0. \tag{20}$$

6 Scale-averaged Floquet closure

Let $u = |1\rangle$ and let

$$\sigma = \operatorname{sgn}\big(\langle m|f(B)|1\rangle\big), \qquad v = \sigma|m\rangle.$$

Then $u \perp v$, $\langle v|f(B)|u\rangle > 0$, and we define

$$C = Buv^\dagger = \sigma B|1\rangle\langle m|. \tag{21}$$

Thus the endpoint value in (20) is positive after replacing the endpoint vector by v. Since B is tridiagonal and $m \geq 3$, $Bu \perp v$. Let

$$Y_\theta = \mathrm{e}^{i\theta} C + \mathrm{e}^{-i\theta} C^\dagger.$$

The nonzero eigenvalues of Y_θ are $\pm\|Bu\|$, so its spectrum is independent of θ and $\|Y_\theta\| = \|Bu\| \leq 1$.

Fix $t \in (0, 1/2)$, put $\rho = 1 - t$, and define

$$J_{s,t}(\theta) = \rho s B + t Y_\theta, \qquad 0 \leq s \leq 1. \tag{22}$$

Then $\|J_{s,t}(\theta)\| \leq \rho s + t \leq 1$. Define

$$\Phi_t(f) = \frac{1}{2\pi t} \int_0^1 \int_0^{2\pi} \mathrm{e}^{-i\theta} \operatorname{Tr} f(J_{s,t}(\theta)) \, \mathrm{d}\theta \, \mathrm{d}s. \tag{23}$$

The coefficient is real because $J_{s,t}(-\theta) = \overline{J_{s,t}(\theta)}$.

6.1 Infinitesimal motivation

For a polynomial p, the first Fourier coefficient of the derivative at $t = 0$ is

$$\operatorname{Tr}[p'(sB)C] = \langle v|p'(sB)B|u\rangle = \left\langle v \,\middle|\, \frac{\mathrm{d}}{\mathrm{d}s} p(sB) \,\middle|\, u \right\rangle.$$

Therefore scale averaging gives

$$\lim_{t \downarrow 0} \Phi_t(p) = \langle v|p(B)|u\rangle.$$

If the factor $\rho = 1 - t$ is differentiated as well, the resulting term is independent of θ and is killed by the first Fourier projection.

6.2 Perturbation and domain lemmas

Lemma 6.1 (Second-order eigenvalue remainder)**.** *Let A be Hermitian with simple eigenvalues separated by $\gamma > 0$, and let $\|E\| \le \gamma/4$. If $Aq_j = \lambda_j q_j$, then the corresponding eigenvalue of $A+E$ satisfies*

$$|\lambda_j(A+E) - \lambda_j - \langle q_j|E|q_j\rangle| \le \frac{4\,\|E\|^2}{\gamma}.$$

A proof is included in Section A.

Let $Bq_j = \lambda_j q_j$ and set

$$\alpha_j = \langle v|q_j\rangle\langle q_j|u\rangle, \qquad c_j = \langle q_j|C|q_j\rangle = \lambda_j\alpha_j. \tag{24}$$

All quantities are real.

Lemma 6.2 (The scalar first-order model remains in the domain)**.** *For every $s \in [0,1]$, $\theta \in \mathbb{R}$, and j,*

$$|\rho s\lambda_j + 2tc_j\cos\theta| \le 1. \tag{25}$$

Proof. Since u and v are orthogonal and q_j is normalized,

$$2\,|\alpha_j| \le |\langle u|q_j\rangle|^2 + |\langle v|q_j\rangle|^2 \le 1.$$

Therefore

$$|\rho s\lambda_j + 2tc_j\cos\theta| \le |\lambda_j|\,(\rho + 2t\,|\alpha_j|) \le |\lambda_j|\,(\rho+t) \le 1.$$

□

6.3 Quantitative closure theorem

Theorem 6.3 (Quantitative scale-averaged closure)**.** *There is an absolute constant C_0 such that, for every K-Lipschitz $f : [-1,1] \to [-1,1]$ and every $t \le g/16$,*

$$|\Phi_t(f) - \langle v|f(B)|u\rangle| \le C_0\frac{(K+1)mt}{g^2}\left(1 + \log\frac{g}{t}\right). \tag{26}$$

Proof. Choose $s_0 \in (0,1)$ so that $t \le \rho s_0 g/4$; later set $s_0 = 4t/(\rho g)$.

Small scales. At $s = 0$, the spectrum of tY_θ is independent of θ, hence its first Fourier coefficient vanishes. Weyl's inequality and scalar Lipschitz continuity imply

$$|\mathrm{Tr}\, f(J_{s,t}(\theta)) - \mathrm{Tr}\, f(J_{0,t}(\theta))| \le Km\rho s.$$

After division by t and integration over $s \in [0, s_0]$, this contributes at most

$$\frac{Km\rho s_0^2}{2t}. \tag{27}$$

The first-order scalar model differs from the model with $c_j = 0$ by at most $2Kt\,|c_j|$. Here

$$\sum_j |c_j| \le \|C\|_* = \|Bu\|\ \|v\| \le 1,$$

where $\|\cdot\|_*$ is the trace norm. Thus its omitted small-s contribution is at most

$$2Ks_0. \tag{28}$$

Large scales: replacing the true eigenvalues. For $s \geq s_0$, the unperturbed matrix $\rho s B$ has gap at least $\rho s g$. By Lemma 6.1, the eigenvalues of $J_{s,t}(\theta)$ can be matched as

$$\mu_j(s,\theta) = \rho s \lambda_j + 2tc_j \cos\theta + r_j(s,\theta), \qquad |r_j(s,\theta)| \leq \frac{4t^2}{\rho s g}.$$

The error in the trace of f is therefore at most $4Kmt^2/(\rho s g)$. After taking the Fourier coefficient, dividing by t, and integrating, this contributes at most

$$\frac{4Kmt}{\rho g} \log \frac{1}{s_0}. \tag{29}$$

Exact integration of the scalar model. By Lemma 6.2, all scalar arguments below lie in $[-1,1]$. For fixed j, set $\beta = \rho\lambda_j$ and $h_\theta = 2tc_j \cos\theta$. If $\lambda_j = 0$, then $c_j = 0$. Otherwise,

$$\int_0^1 f(\beta s + h_\theta)\,\mathrm{d}s = \frac{1}{\beta} \int_{h_\theta}^{\beta + h_\theta} f(x)\,\mathrm{d}x.$$

For any signed h whose integration intervals remain in $[-1,1]$,

$$\int_\beta^{\beta+h} f(x)\,\mathrm{d}x - \int_0^h f(x)\,\mathrm{d}x = h(f(\beta) - f(0)) + R, \qquad |R| \leq Kh^2.$$

Since

$$\frac{1}{2\pi t} \int_0^{2\pi} \mathrm{e}^{-i\theta} h_\theta \,\mathrm{d}\theta = c_j,$$

the main contribution of index j is

$$\frac{c_j}{\rho\lambda_j}\big(f(\rho\lambda_j) - f(0)\big) = \frac{\alpha_j}{\rho}\big(f(\rho\lambda_j) - f(0)\big).$$

The remainder is at most $2Kt\,|\lambda_j|\,\alpha_j^2/\rho$. Since $\sum_j \alpha_j^2 \leq 1$ and $\sum_j \alpha_j = \langle v|u\rangle = 0$, the sum of the main terms is

$$\frac{1}{\rho}\langle v|f(\rho B)|u\rangle,$$

and the total scalar-model remainder is at most $2Kt/\rho$. Because B and ρB commute,

$$\|f(\rho B) - f(B)\| \leq Kt.$$

Using $\|f(B)\| \leq 1$, the radial-rescaling error is at most $(K+1)t/\rho$. Hence the scalar-model and rescaling contribution is bounded by

$$\frac{(3K+1)t}{\rho}. \tag{30}$$

Combining (27), (28), (29), and (30) gives

$$|\Phi_t(f) - \langle v|f(B)|u\rangle| \leq \frac{Km\rho s_0^2}{2t} + 2Ks_0 + \frac{4Kmt}{\rho g} \log \frac{1}{s_0} + \frac{(3K+1)t}{\rho}.$$

Take $s_0 = 4t/(\rho g)$. If $t \leq g/16$, then $s_0 \leq 1/2$ and the perturbation condition holds. Since $\rho \geq 1/2$ and $g \leq 2$, the displayed bound is at most the right-hand side of (26), after increasing an absolute constant C_0. $\square$

6.4 An explicit inverse-polynomial closure parameter

For $0 < x \le 1$, the elementary inequality

$$x(1 + \log(1/x)) \le 2\sqrt{x} \tag{31}$$

holds. Put $x = t/g$. Therefore (26) is at most

$$\frac{2C_0(K+1)m}{g}\sqrt{\frac{t}{g}}.$$

Consequently, the explicit choice

$$t \le \min\left\{\frac{g}{16}, g\left(\frac{\varepsilon_0 g}{8C_0(K+1)m}\right)^2\right\} \tag{32}$$

ensures

$$\Phi_t(f) \ge \frac{\varepsilon_0}{2}. \tag{33}$$

By (16), this t is inverse polynomial whenever K, d are polynomially bounded.

7 Circuit-to-Floquet embedding

Let a source unitary U be implemented by $L \le m-2$ local gates. Insert an identity gate at the first clock edge and write

$$U = G_{m-1}\cdots G_2G_1, \qquad G_1 = I,$$

where unused positions contain identity gates. Let $V_1 = I$ and $V_j = G_{j-1}\cdots G_1$. Define the scaled open history Hamiltonian using the entries of B,

$$H_{\text{open}}(s) = \rho s\sum_{j=1}^{m} a_j|j\rangle\langle j|\otimes I \tag{34}$$

$$+\rho s\sum_{j=1}^{m-1} b_j\left(|j+1\rangle\langle j|\otimes G_j + |j\rangle\langle j+1|\otimes G_j^\dagger\right). \tag{35}$$

Since $C = \sigma B|1\rangle\langle m| = \sigma(a_1|1\rangle\langle m| + b_1|2\rangle\langle m|)$, add

$$H_{\text{close}}(t) = t\sigma a_1(|1\rangle\langle m| + |m\rangle\langle 1|)\otimes I \tag{36}$$

$$+ t\sigma b_1(|2\rangle\langle m| + |m\rangle\langle 2|)\otimes I, \tag{37}$$

The factor σ appears only in the closing edges. Put $H_{s,t}(U) = H_{\text{open}}(s) + H_{\text{close}}(t)$.

Let $W = \sum_j |j\rangle\langle j|\otimes V_j$. Then $W^\dagger H_{\text{open}}(s)W = \rho sB\otimes I$. Moreover, since $V_1 = V_2 = I$ and $V_m = U$,

$$W^\dagger H_{\text{close}}(t)W = t(C\otimes U + C^\dagger\otimes U^\dagger).$$

Thus each eigenphase $e^{i\theta}$ of U gives precisely the fiber $J_{s,t}(\theta)$, and

$$\operatorname{Tr} f(H_{s,t}(U)) = \sum_{e^{i\theta}\in\operatorname{spec}(U)} \operatorname{Tr} f(J_{s,t}(\theta)). \tag{38}$$

Each matrix unit on the clock acts on $O(\log m)$ qubits and each G_j acts on $O(\log n)$ qubits. Hence the Hamiltonian is log-local, has polynomially many terms, and has norm at most one.

8 Positive phase averaging and a single packed instance

This section removes the need to know the exact value of $\Phi_t(f)$ and avoids signed Turing queries.

8.1 A positive continuous average

For $U_\phi = \mathrm{e}^{i\phi}U$, define
$$Y_U(s,\phi) = \frac{1}{mN}\operatorname{Tr} f(H_{s,t}(U_\phi)).$$
Let
$$p(\phi) = \frac{1+\cos\phi}{2\pi} \ge 0. \tag{39}$$
For fixed s, write
$$F_s(\theta) = \operatorname{Tr} f(J_{s,t}(\theta)) = \sum_{k\in\mathbb{Z}} a_k(s)\mathrm{e}^{ik\theta}.$$
The function is real and even, so $a_{-k} = a_k \in \mathbb{R}$. The density in (39) has only Fourier coefficients $0, \pm 1$, with the two nonzero first coefficients equal to $1/2$. Therefore
$$\mathcal{A}(U) := \int_0^1 \int_0^{2\pi} p(\phi) Y_U(s,\phi)\,\mathrm{d}\phi\,\mathrm{d}s = B_f + \frac{t\Phi_t(f)}{m}\operatorname{Re}\tau(U), \tag{40}$$
where the U-independent baseline is
$$B_f = \frac{1}{m}\int_0^1 a_0(s)\,\mathrm{d}s = \frac{1}{2\pi m}\int_0^1\int_0^{2\pi} \operatorname{Tr} f(J_{s,t}(\theta))\,\mathrm{d}\theta\,\mathrm{d}s. \tag{41}$$
By (33), the signal coefficient is positive and satisfies
$$\frac{t\Phi_t(f)}{m} \ge \frac{t\varepsilon_0}{2m}. \tag{42}$$
No division by the unknown exact value of $\Phi_t(f)$ is needed for a sign-promise reduction.

8.2 Uniform Lipschitz bounds and quadrature

Lemma 8.1 (Lipschitz bounds for the normalized trace)**.** *For all $s, s' \in [0,1]$ and phases ϕ, ϕ',*
$$\left|Y_U(s,\phi) - Y_U(s',\phi)\right| \le K\left|s - s'\right|, \tag{43}$$
$$\left|Y_U(s,\phi) - Y_U(s,\phi')\right| \le 2Kt\left|\phi - \phi'\right|. \tag{44}$$
Consequently, $p(\phi)Y_U(s,\phi)$ has a uniform polynomial Lipschitz constant, and (40) can be approximated to error ξ by a rectangular grid of size $\operatorname{poly}(K, 1/t, 1/\xi)$, uniformly for every source unitary U.

Proof. For Hermitian matrices of the same dimension, matching ordered eigenvalues and using Weyl's inequality gives
$$\left|\frac{1}{L}\operatorname{Tr} f(A) - \frac{1}{L}\operatorname{Tr} f(A')\right| \le K\left\|A - A'\right\|.$$
The scale difference has norm at most $|s-s'|$. The phase appears only in $t(C\otimes \mathrm{e}^{i\phi}U + C^\dagger \otimes \mathrm{e}^{-i\phi}U^\dagger)$, so the norm difference is at most $2t\,|\phi-\phi'|$. Since p and p' are bounded by absolute constants and $|Y_U| \le 1$, standard Riemann-sum error bounds give the final statement. □

The grid probabilities can be approximated in total variation by nonnegative rational weights with a common polynomial denominator. Since $|Y_U| \le 1$, total variation error χ changes the average by at most 2χ.

8.3 Classical computation of the baseline

Lemma 8.2 (The baseline is efficiently computable)**.** *Under Assumption 2.1, B_f in (41) can be computed to additive error ξ in time polynomial in $m, K, 1/t, 1/\xi$.*

Proof. Use the same polynomial grid as in Lemma 8.1. At each grid point, $J_{s,t}(\theta)$ is an explicit $m \times m$ Hermitian matrix. Its eigenvalues can be computed to polynomial precision by standard polynomial-time numerical linear algebra, and f can be evaluated to the required precision by Assumption 2.1. Lipschitz continuity controls the propagation of eigenvalue errors to the trace. $\square$

8.4 Selector packing

Let $\{(s_\ell, \phi_\ell), w_\ell\}_{\ell=1}^{R}$ be a rational positive quadrature rule, $w_\ell = k_\ell/P$, $\sum_\ell k_\ell = P$, with $R, P = \mathrm{poly}(n)$. Form the direct sum containing k_ℓ copies of the qubit-encoded Hamiltonian $\widehat{H}_{s_\ell,t}(U_{\phi_\ell})$. Encode the copy index in $q_{\mathrm{sel}} = \lceil \log_2 P \rceil$ selector qubits, and put the zero Hamiltonian on unused selector states. Denote the resulting Hamiltonian by $\mathcal{H}_U$.

The selector projectors act on $O(\log P)$ qubits; therefore $\mathcal{H}_U$ remains log-local and has polynomially many terms. Let

$$\alpha = \frac{m}{2^{\lceil \log_2 m \rceil}} > \frac{1}{2}, \qquad \beta = \frac{P}{2^{\lceil \log_2 P \rceil}} > \frac{1}{2}.$$

Combining clock padding, selector padding, and the uniform quadrature error gives

$$\frac{\mathrm{Tr}\, f(\mathcal{H}_U)}{\dim \mathcal{H}_U} = B_{\mathrm{pack}} + A_{\mathrm{pack}} \,\mathrm{Re}\, \tau(U) + e_U, \qquad |e_U| \le \xi, \tag{45}$$

where B_{pack} is classically computable and

$$A_{\mathrm{pack}} = \alpha\beta \frac{t\Phi_t(f)}{m} = \beta \frac{t\Phi_t(f)}{2^{\lceil \log_2 m \rceil}} \ge \frac{t\varepsilon_0}{8m}. \tag{46}$$

The known $f(0)$ contributions from invalid clock and selector states are included in B_{pack}.

9 Complete Lipschitz hardness theorem

Theorem 9.1 (Finite-size Lipschitz reduction)**.** *Let $f : [-1,1] \to [-1,1]$ be K-Lipschitz and let $d = \widetilde{\deg}_{\varepsilon_0}(f)$. Suppose the Jacobi witness can be implemented to the precision in (18). Let U be an n-qubit unitary implemented by at most $d-1$ local gates, and suppose*

$$\mathrm{Re}\, \tau(U) \ge \delta \quad \text{or} \quad \mathrm{Re}\, \tau(U) \le -\delta.$$

Choose t as in (32). Then in polynomial time one can construct a single log-local Hamiltonian $\mathcal{H}_U$, with norm at most one, and a classically computable baseline B_{pack}, such that the two promises imply respectively

$$\frac{\mathrm{Tr}\, f(\mathcal{H}_U)}{\dim \mathcal{H}_U} - B_{\mathrm{pack}} \ge \frac{t\varepsilon_0\delta}{16m},$$

and

$$\frac{\mathrm{Tr}\, f(\mathcal{H}_U)}{\dim \mathcal{H}_U} - B_{\mathrm{pack}} \le -\frac{t\varepsilon_0\delta}{16m}.$$

It is enough to estimate the normalized target trace, the baseline, and the packing error each to accuracy at most

$$\eta_{\mathrm{tr}} \le \frac{t\varepsilon_0\delta}{64m}. \tag{47}$$

If $K, d, 1/\delta$ are polynomially bounded, then $1/t$, the number of Hamiltonian terms, and $1/\eta_{\mathrm{tr}}$ are polynomially bounded. No condition on E_d/E_{d-1} is used.

Proof. Use Proposition 4.2 and Corollary 5.2 to obtain the oriented rational Jacobi witness with constant endpoint signal and inverse-polynomial gap. The choice (32) and Theorem 6.3 imply (33). Embed the source circuit by Equations (34) and (36). Apply the positive phase average (40), approximate it by the positive rational quadrature from Lemma 8.1, and pack the instances into $\mathcal{H}_U$. Choose the combined quadrature and rational-weight error to be at most $t\varepsilon_0\delta/(64m)$. By (46), the true signal has magnitude at least $t\varepsilon_0\delta/(8m)$, so the stated error budget leaves the displayed promise gap. The baseline is computable to the same accuracy by Lemma 8.2. All dimensions, grids, rational denominators, and coefficient precisions are polynomial because $m = d + 1$, K, $1/g$, $1/t$, and $1/\delta$ are polynomial. □

Definition 9.2 (Size-compatible function family)**.** A uniformly effective family $\{f_s\}$ is called size-compatible with the DQC1 source problem if there is a polynomial-time computable polynomial padding map $s = s(n)$ such that, for the chosen DQC1-complete circuit family of size n and gate count $L(n)$,

(i) $d_s := \widetilde{\deg}_{\varepsilon_0}(f_s) \geq L(n) + 1$;

(ii) d_s and $K_s := \mathrm{Lip}(f_s)$ are bounded by $\mathrm{poly}(n)$;

(iii) s is large enough to contain the source work register, clock, selector, and any unused identity qubits.

Condition (iii) causes no change in the normalized trace: tensoring the constructed Hamiltonian with an identity register preserves $\operatorname{Tr} f(H)/\dim(H)$. Likewise, replacing U by $U \otimes I$ preserves its normalized trace.

Corollary 9.3 (Uniform DQC1-hardness without the successive-error condition)**.** *Let $\{f_s\}$ be a size-compatible family satisfying Assumption 2.1, with $K_s = \mathrm{poly}(s)$ and constant-error approximate degree growing sufficiently rapidly to contain the padded source circuits. Then estimating*

$$\frac{\operatorname{Tr} f_s(A)}{2^s}$$

for log-local Hermitian A with $\|A\| \leq 1$, to a sufficiently small inverse-polynomial additive error, is DQC1-hard under a polynomial-time single-instance metric reduction. No condition on $E_{d_s}(f_s)/E_{d_s-1}(f_s)$ is required.

Proof. Apply Theorem 9.1 with $f = f_{s(n)}$, then pad the target Hamiltonian by identity registers to exactly $s(n)$ qubits. The source normalized-trace promise problem is DQC1-complete, and the target accuracy in (47) is inverse polynomial. □

Corollary 9.4 (DQC1-completeness under the membership hypothesis)**.** *Under the hypotheses of Corollary 9.3, the problem is DQC1-complete. Membership follows from the DQC1 phase-estimation algorithm for log-local Hamiltonians and polynomially Lipschitz functions [3, Proposition 2.3].*

10 Comparison of the two routes

	Uniform Chebyshev clock	Lipschitz f-dependent Floquet clock
Clock parameters	$a_j = 0$, $b_j = 1/2$; the selected length depends on f	Jacobi entries depend on an equioscillation dual witness
Extra function hypothesis	Polynomial-size D with sufficiently small E_D	Polynomial Lipschitz constant and effective witness construction
Quantitative condition	$(D - d + 1)(E_D + \eta_{\rm tr}) \lesssim \varepsilon_0 \delta_U$	$\eta_{\rm tr} \lesssim t\varepsilon_0\delta_U/m$, with explicit $t = 1/\operatorname{poly}$
Signal extraction	Discrete Fourier transform over phases	Positive phase average and one selector-packed Hamiltonian
$E_d/E_{d-1} < 1/2$	Not used	Not used
Reduction	Polynomially many phase instances, or standard packing	Single-instance metric reduction after positive packing

11 Limitations and remaining questions

(1) The uniform-clock theorem is conditional on a high-accuracy polynomial approximant. It is not automatic for an arbitrary continuous function.

(2) The Lipschitz theorem removes that approximation condition, but uniform hardness still requires an effective representation of the function and its Jacobi witness. The exact finite-dimensional witness exists without this computational assumption.

(3) The Floquet fiber has two finite-rank closing edges when the first diagonal entry of B is nonzero. It is not necessarily a strict scalar periodic Jacobi matrix.

(4) A natural inverse-spectral question is whether the same endpoint functional can always be realized by a strict scalar periodic Jacobi family with only polynomial loss.

A Proof of the eigenvalue perturbation lemma

Let $Aq_j = \lambda_j q_j$, $P = |q_j\rangle\langle q_j|$, and $Q = I - P$. Weyl's inequality gives $|\mu - \lambda_j| \le \|E\|$. Write a corresponding eigenvector as $\psi = aq_j + \chi$, where $\chi \in \operatorname{ran} Q$ and $a \neq 0$. The Q-projected eigenvalue equation can be written

$$Q(A + E - \mu)Q\,\frac{\chi}{a} = -QEq_j.$$

On $\operatorname{ran} Q$, the operator $Q(A - \lambda_j)Q$ has inverse norm at most $1/\gamma$. The perturbation from it to $Q(A + E - \mu)Q$ has norm at most $\|E\| + |\mu - \lambda_j| \le 2\|E\| \le \gamma/2$. Hence a Neumann-series argument gives

$$\left\|\frac{\chi}{a}\right\| \le \frac{2\|E\|}{\gamma}.$$

The P-projected equation now yields the exact identity

$$\mu - \lambda_j - \langle q_j|E|q_j\rangle = \left\langle q_j \middle| E \middle| \frac{\chi}{a} \right\rangle.$$

Its absolute value is at most $2\|E\|^2/\gamma$, and therefore certainly at most $4\|E\|^2/\gamma$. This proves Lemma 6.1.

B A concrete quadrature budget

Let $G_U(s,\phi) = p(\phi)Y_U(s,\phi)$. Since $\|p\|_\infty \le 1/\pi$, $\|p'\|_\infty \le 1/(2\pi)$, and Lemma 8.1 holds, there are absolute constants L_s, L_ϕ with

$$|G_U(s,\phi) - G_U(s',\phi')| \le L_s K|s-s'| + L_\phi(1+Kt)|\phi-\phi'|.$$

A midpoint grid with spacings

$$\Delta s \le \frac{\xi}{8L_s K}, \qquad \Delta\phi \le \frac{\xi}{8L_\phi(1+Kt)}$$

has integration error at most $\xi/2$. Approximate its positive normalized weights in total variation by at most $\xi/4$; the induced error is at most $\xi/2$ because $|Y_U| \le 1$. The common denominator and the number of grid points are polynomial in K, $1/t$, and $1/\xi$. This proves the quantitative packing claim used in Theorem 9.1.

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