Two Routes Beyond the Successive-Error Gap in DQC1 Normalized Trace Estimation

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Abstract

We study two mechanisms for removing the successive-best-approximation condition $E_d/E_{d-1} < 1/2$ from reductions proving DQC1-hardness of normalized trace estimation $\mathrm{Tr}[f(A)]/\dim(A)$ for log-local Hamiltonians. The first mechanism uses a high-accuracy polynomial P_D and a uniform Chebyshev clock. It gives a simple sufficient condition

$$(D - d + 1)(E_D + \eta_{\mathrm{tr}}) \le c\varepsilon_0,$$

where d is the ε_0-approximate degree and η_{tr} is the normalized-trace estimation error. The second mechanism applies to polynomially Lipschitz functions and does not require any higher-degree approximation P_D. An approximate-degree dual witness is realized as an endpoint matrix element of an f-dependent open Jacobi matrix. A scale-averaged finite-rank Floquet closure converts this endpoint functional into the first Floquet harmonic of a trace. The conversion has only inverse-polynomial loss, and therefore inverse-polynomial-accuracy normalized trace estimation is DQC1-hard without the condition $E_d/E_{d-1} < 1/2$. Detailed proofs, parameter dependencies, and the circuit-to-Hamiltonian embedding are given.

Scope of the note

The first result below is an elementary polynomial/Fourier reduction. The second result is stated under the standard effective-construction assumption that the alternation points and the associated Jacobi coefficients can be computed to polynomially many bits. The hardness reduction is presented as a polynomial-time classical Turing reduction. The Floquet clock in the second result is a finite-rank closure of an open Jacobi chain; it is not, in general, a strict scalar periodic Jacobi matrix.

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1 Introduction

Let $f : [-1, 1] \to [-1, 1]$ be continuous and let

$$E_k(f) := \inf_{p \in \mathcal{P}_k} \|f - p\|_\infty, \qquad \widetilde{\deg}_\varepsilon(f) := \min\{k : E_k(f) \le \varepsilon\}.$$

The work of Ji, Li, Shao, Wang, and Zhang [4] proves DQC1-completeness of normalized trace estimation for functions of log-local Hamiltonians under the technical condition

$$\frac{E_d(f)}{E_{d-1}(f)} < \frac{1}{2}, \qquad d = \widetilde{\deg}_{\varepsilon_0}(f),$$

for a constant ε_0. The condition enters because their periodic-Jacobi discriminant produces a signal proportional to $E_{d-1} - E_d$, while the residual error is controlled by E_d.

This note develops two ways to avoid that comparison.

(i) A *uniform-clock route*: a high-accuracy degree-D approximation must have a non-negligible Chebyshev coefficient in degrees $d, \dots, D$. A uniform cycle of the matching length isolates this coefficient exactly.

(ii) An *f-dependent Lipschitz route*: the alternation measure of best approximation is realized as an endpoint matrix element of an open Jacobi matrix, following the philosophy of Montanaro and Shao [7]. A radial or scale average converts the derivative created by closing the chain into the original matrix function.

The two results have different strengths. The uniform-clock route is exceptionally simple, but requires a degree-D approximation whose error decreases fast enough relative to $D - d$. The Lipschitz route removes this additional approximation hypothesis but uses an f-dependent clock and a finite-rank Floquet closure.

2 Preliminaries

2.1 Approximation and normalized traces

Throughout, $\mathcal{P}_k$ denotes the real polynomials of degree at most k. Fix a constant $\varepsilon_0 \in (0, 1)$ and put

$$d := \widetilde{\deg}_{\varepsilon_0}(f).$$

Then

$$E_{d-1}(f) > \varepsilon_0, \qquad E_d(f) \le \varepsilon_0. \tag{1}$$

We use the Chebyshev polynomials $T_k(\cos\vartheta) = \cos(k\vartheta)$.

For an N-dimensional unitary U, write

$$\tau(U) := \frac{\operatorname{Tr} U}{N}.$$

Estimating $\operatorname{Re}\tau(U)$ to any promised accuracy in $[1/\operatorname{poly}(n), 1)$ is DQC1-complete [6, 4].

2.2 A circuit-fiber identity

Let $U = G_{m-1} \cdots G_1$ be a circuit and put $V_1 = I$, $V_j = G_{j-1} \cdots G_1$ for $j \ge 2$, so that $V_m = U$. For real a_j and positive b_j, the open history Hamiltonian is

$$H_{\text{open}} = \sum_{j=1}^{m} a_j |j\rangle\langle j| \otimes I + \sum_{j=1}^{m-1} b_j \left(|j+1\rangle\langle j| \otimes G_j + |j\rangle\langle j+1| \otimes G_j^\dagger\right). \tag{2}$$

The gauge transformation

$$W = \sum_{j=1}^{m} |j\rangle\langle j| \otimes V_j$$

turns H_{open} into $B \otimes I$, where B is the Jacobi matrix with diagonal a_j and subdiagonal b_j. Closing the clock by terms whose gauge transforms contain U therefore decomposes the Hamiltonian into fibers indexed by eigenphases of U. This is the basic circuit-to-periodic-Jacobi mechanism used in [4].

3 A uniform Chebyshev clock under a high-accuracy approximation

This section gives the first sufficient condition. It does not require an f-dependent Jacobi matrix.

3.1 A large coefficient in the Chebyshev tail

Lemma 3.1 (Chebyshev-tail certificate)**.** *Let $d = \widetilde{\deg}_{\varepsilon_0}(f)$. Suppose*

$$P_D(x) = \sum_{k=0}^{D} c_k T_k(x), \qquad \|f - P_D\|_\infty \le E_D, \qquad D \ge d.$$

Then

$$\sum_{k=d}^{D} |c_k| > \varepsilon_0 - E_D. \tag{3}$$

Consequently, there is an $r \in \{d, \dots, D\}$ such that

$$|c_r| \ge \frac{\varepsilon_0 - E_D}{D - d + 1}. \tag{4}$$

Proof. Set $Q_{d-1} = \sum_{k=0}^{d-1} c_k T_k$. Since $Q_{d-1} \in \mathcal{P}_{d-1}$, (1) implies $\|f - Q_{d-1}\|_\infty > \varepsilon_0$. On the other hand, using $\|T_k\|_\infty = 1$,

$$\|f - Q_{d-1}\|_\infty \le \|f - P_D\|_\infty + \left\| \sum_{k=d}^{D} c_k T_k \right\|_\infty \le E_D + \sum_{k=d}^{D} |c_k| \,.$$

This proves (3); the pigeonhole principle gives (4). □

3.2 The uniform clock as an exact frequency filter

For $r \ge 3$, let

$$J_r(\theta) = \frac{1}{2} \begin{pmatrix} 0 & 1 & & & \mathrm{e}^{-i\theta} \\ 1 & 0 & 1 & & \\ & 1 & 0 & \ddots & \\ & & \ddots & \ddots & 1 \\ \mathrm{e}^{i\theta} & & & 1 & 0 \end{pmatrix} \in \mathbb{C}^{r \times r}. \tag{5}$$

Its eigenvalues are

$$\lambda_\ell(\theta) = \cos\left(\frac{\theta + 2\pi\ell}{r}\right), \qquad \ell = 0, \dots, r-1.$$

Lemma 3.2 (Exact filtering)**.** *For every $k \geq 0$,*

$$\frac{1}{r}\operatorname{Tr} T_k(J_r(\theta)) = \begin{cases} \cos(q\theta), & k = qr, \\ 0, & r \nmid k. \end{cases} \tag{6}$$

Consequently,

$$\frac{1}{r}\operatorname{Tr} P_D(J_r(\theta)) = \sum_{q=0}^{\lfloor D/r \rfloor} c_{qr}\cos(q\theta). \tag{7}$$

Proof. Using $T_k(\cos x) = \cos(kx)$,

$$\operatorname{Tr} T_k(J_r(\theta)) = \operatorname{Re}\left[\mathrm{e}^{ik\theta/r}\sum_{\ell=0}^{r-1}\mathrm{e}^{2\pi ik\ell/r}\right].$$

The geometric sum is zero unless $r \mid k$, and equals r when $k = qr$. □

3.3 Phase filtering and the quantitative reduction

Let $H_r(U)$ be the uniform cyclic history Hamiltonian obtained by placing a circuit for U on an r-site clock, padding with identity gates if necessary. If the eigenvalues of U are $\mathrm{e}^{i\theta_j}$, then

$$H_r(U) \simeq \bigoplus_j J_r(\theta_j).$$

For a phase ϕ, define $U_\phi = \mathrm{e}^{i\phi}U$ and

$$Y(\phi) := \frac{1}{rN}\operatorname{Tr} f(H_r(U_\phi)).$$

By Theorem 3.2,

$$Y(\phi) = c_0 + \sum_{q=1}^{Q} c_{qr}\operatorname{Re}\left(\mathrm{e}^{iq\phi}\tau(U^q)\right) + \rho(\phi), \qquad Q = \left\lfloor \frac{D}{r} \right\rfloor, \tag{8}$$

where $|\rho(\phi)| \leq E_D$.

Take $M = 2Q+1$ equally spaced phases $\phi_s = 2\pi s/M$. The discrete Fourier transform has no aliasing among the frequencies $-Q, \dots, Q$, and hence

$$\frac{1}{M}\sum_{s=0}^{M-1}\mathrm{e}^{-i\phi_s}Y(\phi_s) = \frac{c_r}{2}\tau(U) + \widehat{\rho}_1, \qquad |\widehat{\rho}_1| \leq E_D. \tag{9}$$

If every value $Y(\phi_s)$ is estimated to additive error η_{tr}, the additional Fourier error is at most η_{tr}.

Theorem 3.3 (Uniform-clock sufficient condition)**.** *Let $d = \widetilde{\deg}_{\varepsilon_0}(f)$ and let P_D be as in Theorem 3.1. Assume that the corresponding coefficient c_r and the clock can be constructed with polynomially many bits. Then a normalized-trace oracle of accuracy η_{tr} allows one to recover $\tau(U)$ with error at most*

$$\delta_U \leq \frac{2(E_D + \eta_{\mathrm{tr}})}{|c_r|} \leq \frac{2(D-d+1)(E_D+\eta_{\mathrm{tr}})}{\varepsilon_0 - E_D}. \tag{10}$$

In particular, if $E_D \leq \varepsilon_0/2$ and

$$(D-d+1)(E_D+\eta_{\mathrm{tr}}) \leq c\varepsilon_0 \tag{11}$$

for a sufficiently small absolute constant c, then constant-accuracy estimation of $\tau(U)$ reduces to normalized trace estimation of $f(H_r(U_\phi))$. More generally, to obtain target accuracy δ_U, it suffices that

$$(D-d+1)(E_D+\eta_{\mathrm{tr}}) \leq \frac{\delta_U}{4}\varepsilon_0. \tag{12}$$

Proof. Use the coefficient r from Theorem 3.1, construct $H_r(U_\phi)$, and apply (9). Dividing the total Fourier error $E_D + \eta_{\rm tr}$ by $|c_r|/2$ gives the first inequality in (10). The second follows from (4). The stated sufficient conditions are immediate. $\square$

Remark 3.4. The condition in (11) removes the ratio E_d/E_{d-1} but replaces it by a vanishing-error approximability condition. For a completely arbitrary continuous function, there need not be a polynomially bounded D for which E_D is inverse polynomial.

4 An f-dependent Lipschitz construction without the successive-error gap

We now avoid the higher-degree polynomial P_D entirely. The argument has four components: (i) an open Jacobi endpoint witness; (ii) inverse-polynomial spectral separation from Lipschitz regularity; (iii) a scale-averaged Floquet closure; and (iv) a history-Hamiltonian embedding.

4.1 An open Jacobi witness from equioscillation

Let $p^* \in \mathcal{P}_{d-1}$ be a best approximant and put $E = E_{d-1}(f) > \varepsilon_0$. By the Chebyshev equioscillation theorem [2, 9], there are $m = d+1$ points

$$-1 \le x_1 < x_2 < \cdots < x_m \le 1$$

and a sign σ such that

$$f(x_j) - p^*(x_j) = \sigma(-1)^j E. \tag{13}$$

Let

$$\omega(x) = \prod_{j=1}^{m}(x - x_j), \qquad h_j = \frac{\alpha}{\omega'(x_j)}, \qquad \sum_{j=1}^{m} |h_j| = 1,$$

where α is chosen with the sign that makes $h_j(f(x_j) - p^*(x_j)) \ge 0$.

Lemma 4.1 (Discrete dual witness)**.** *The weights h_j satisfy*

$$\sum_{j=1}^{m} h_j q(x_j) = 0 \quad \textit{for every } q \in \mathcal{P}_{m-2}, \tag{14}$$

and

$$\sum_{j=1}^{m} h_j f(x_j) = E. \tag{15}$$

Proof. The Lagrange interpolation formula implies

$$\sum_{j=1}^{m} \frac{q(x_j)}{\omega'(x_j)} = 0, \qquad \deg q \le m-2,$$

by comparing the coefficient of x^{m-1}. This gives (14). Since $p^* \in \mathcal{P}_{m-2}$, its contribution vanishes. The signs of $1/\omega'(x_j)$ alternate, and therefore (13) and the normalization $\sum_j |h_j| = 1$ give (15). $\square$

Set $w_j = |h_j|$, $D_x = \mathrm{diag}(x_1, \ldots, x_m)$,

$$z = (\sqrt{w_1}, \ldots, \sqrt{w_m})^T, \qquad y = (\mathrm{sgn}(h_1)\sqrt{w_1}, \ldots, \mathrm{sgn}(h_m)\sqrt{w_m})^T.$$

Apply Lanczos orthogonalization to $z, D_x z, \dots, D_x^{m-1} z$. It produces an orthogonal matrix Q for which

$$B = Q^T D_x Q = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{m-1} \\ & & & b_{m-1} & a_m \end{pmatrix}, \qquad b_j > 0. \tag{16}$$

The first Lanczos vector is z. By (14), y is orthogonal to $z, D_x z, \dots, D_x^{m-2} z$, so the last Lanczos vector is $\pm y$.

Proposition 4.2 (Open Jacobi endpoint witness)**.** *The Jacobi matrix B in* (16) *satisfies $\|B\| \le 1$ and*

$$|\langle m|f(B)|1\rangle| = E_{d-1}(f) > \varepsilon_0. \tag{17}$$

Proof. Since B is orthogonally similar to D_x, its spectrum is contained in $[-1, 1]$. Moreover,

$$\langle m|f(B)|1\rangle = \pm y^T f(D_x) z = \pm \sum_{j=1}^{m} h_j f(x_j),$$

and the conclusion follows from Theorem 4.1. □

This is a general-Jacobi version of the tridiagonal dual-witness mechanism in Montanaro and Shao [7, Theorem 1.12].

4.2 Lipschitz regularity gives separated witness eigenvalues

Assume now that f is K-Lipschitz. Since $\|f\|_\infty \le 1$ and the zero polynomial is admissible, $E \le 1$ and

$$\|p^*\|_\infty \le \|f\|_\infty + E \le 2.$$

By Markov's inequality,

$$\left\|(p^*)'\right\|_\infty \le 2(d-1)^2 \le 2d^2. \tag{18}$$

At adjacent alternation points,

$$2E \le |f(x_{j+1}) - f(x_j)| + |p^*(x_{j+1}) - p^*(x_j)|.$$

Hence

$$g := \min_j (x_{j+1} - x_j) \ge \frac{2E}{K + 2d^2} > \frac{2\varepsilon_0}{K + 2d^2}. \tag{19}$$

Thus, if K and d are polynomially bounded, the simple spectrum of B has inverse-polynomial minimum gap.

4.3 Scale-averaged Floquet closure

Let $u = |1\rangle$, $v = |m\rangle$, and define the rank-one operator

$$C = Buv^\dagger = B|1\rangle\langle m|. \tag{20}$$

Since $B|1\rangle = a_1|1\rangle + b_1|2\rangle$ and $m \ge 3$, we have $B|1\rangle \perp |m\rangle$. Consequently,

$$Y_\theta := \mathrm{e}^{i\theta} C + \mathrm{e}^{-i\theta} C^\dagger$$

has norm $\|Y_\theta\| = \|B|1\rangle\| \le 1$, and its spectrum is independent of θ.

Fix $t \in (0, 1/2)$ and put $\rho = 1 - t$. For $s \in [0, 1]$, define

$$J_{s,t}(\theta) = \rho s B + t Y_\theta. \tag{21}$$

Then $\|J_{s,t}(\theta)\| \le \rho s + t \le 1$. Define the scale-averaged first Floquet coefficient

$$\Phi_t(f) := \frac{1}{2\pi t} \int_0^1 \int_0^{2\pi} \mathrm{e}^{-i\theta} \operatorname{Tr} f(J_{s,t}(\theta)) \, \mathrm{d}\theta \, \mathrm{d}s. \tag{22}$$

Because B and C are real, the integrand has the conjugation symmetry $J_{s,t}(-\theta) = \overline{J_{s,t}(\theta)}$; thus $\Phi_t(f)$ is real.

4.3.1 The exact infinitesimal identity

For a polynomial p, differentiation under the trace gives

$$\frac{\partial}{\partial t} \operatorname{Tr} p(sB + tY_\theta)\bigg|_{t=0} = \operatorname{Tr}[p'(sB)Y_\theta].$$

Extracting the first Fourier mode leaves $\operatorname{Tr}[p'(sB)C]$, and

$$\operatorname{Tr}[p'(sB)C] = \langle v | p'(sB)B | u \rangle = \left\langle v \left| \frac{\mathrm{d}}{\mathrm{d}s} p(sB) \right| u \right\rangle.$$

Therefore

$$\lim_{t \downarrow 0} \Phi_t(p) = \langle v | p(B) | u \rangle. \tag{23}$$

The scale integral is precisely what integrates the derivative back to the original function.

4.3.2 A finite-t estimate for Lipschitz functions

We use the following standard simple-eigenvalue perturbation estimate; a proof is recalled in Section A.

Lemma 4.3 (Second-order eigenvalue remainder)**.** *Let A be Hermitian with simple eigenvalues separated by at least $\gamma > 0$, and let $\|E\| \le \gamma/4$. If $\lambda_j(A)$ and q_j are an eigenvalue and normalized eigenvector, then the corresponding eigenvalue of $A + E$ obeys*

$$|\lambda_j(A + E) - \lambda_j(A) - \langle q_j | E | q_j \rangle| \le \frac{4\,\|E\|^2}{\gamma}. \tag{24}$$

Let $Bq_j = \lambda_j q_j$ and set

$$\alpha_j = \langle v | q_j \rangle \langle q_j | u \rangle, \qquad c_j = \langle q_j | C | q_j \rangle = \lambda_j \alpha_j. \tag{25}$$

The quantities are real. Choose $s_0 \in (0, 1)$ with

$$t \le \frac{\rho s_0 g}{4}. \tag{26}$$

For $s \ge s_0$, Theorem 4.3 gives eigenvalues

$$\mu_j(s, \theta) = \rho s \lambda_j + 2tc_j \cos\theta + r_j(s, \theta), \qquad |r_j(s, \theta)| \le \frac{4t^2}{\rho s g}. \tag{27}$$

Lemma 4.4 (Quantitative scale-averaged closure)**.** *Let $f : [-1, 1] \to [-1, 1]$ be K-Lipschitz and let B be as above, with dimension m and minimum eigenvalue gap g. There is a universal constant C_0 such that, whenever $t \le g/16$,*

$$|\Phi_t(f) - \langle v | f(B) | u \rangle| \le C_0 \frac{(K + 1)mt}{g^2} \left(1 + \log \frac{g}{t}\right). \tag{28}$$

Proof. We give the details because this is the central new estimate.

1. The small-s region. At $s = 0$, $J_{0,t}(\theta) = tY_\theta$ has θ-independent spectrum, so its first Fourier coefficient vanishes. Weyl's inequality and the Lipschitz property give

$$|\operatorname{Tr} f(J_{s,t}(\theta)) - \operatorname{Tr} f(J_{0,t}(\theta))| \le Km\rho s.$$

After division by t and integration over $s \in [0, s_0]$, the actual small-s contribution is at most

$$\frac{Km\rho s_0^2}{2t}. \tag{29}$$

The first-order scalar model at small s differs from its value at $c_j = 0$ by at most $2Kt|c_j|$. Since $\sum_j |c_j| \le \|C\|_1 \le 1$, its contribution on $[0, s_0]$ is at most

$$2Ks_0. \tag{30}$$

2. Replacing the true spectrum by the first-order model for $s \ge s_0$. By (27), Lipschitz continuity bounds the trace error by $4Kmt^2/(\rho s g)$. Dividing by t and integrating yields

$$\frac{4Kmt}{\rho g} \log \frac{1}{s_0}. \tag{31}$$

3. Integrating the scalar first-order model. For fixed j, write $\beta = \rho\lambda_j$ and $h_\theta = 2tc_j \cos\theta$. If $\lambda_j = 0$, then $c_j = 0$ and the contribution vanishes. Otherwise,

$$\int_0^1 f(\beta s + h_\theta)\,\mathrm{d}s = \frac{1}{\beta} \int_{h_\theta}^{\beta + h_\theta} f(x)\,\mathrm{d}x.$$

For a K-Lipschitz function,

$$\int_\beta^{\beta+h} f(x)\,\mathrm{d}x - \int_0^h f(x)\,\mathrm{d}x = h(f(\beta) - f(0)) + R, \qquad |R| \le Kh^2.$$

The first Fourier coefficient of h_θ/t is c_j, so the main term is

$$\frac{c_j}{\rho\lambda_j}\big(f(\rho\lambda_j) - f(0)\big) = \frac{\alpha_j}{\rho}\big(f(\rho\lambda_j) - f(0)\big).$$

The total scalar-model remainder is at most

$$\frac{2Kt}{\rho} \sum_j |\lambda_j|\, \alpha_j^2 \le \frac{2Kt}{\rho}.$$

Since $\sum_j \alpha_j = \langle v|u\rangle = 0$, the sum of the main terms equals

$$\frac{1}{\rho}\langle v|f(\rho B)|u\rangle.$$

Finally,

$$\left|\frac{1}{\rho}\langle v|f(\rho B)|u\rangle - \langle v|f(B)|u\rangle\right| \le \frac{(K+1)t}{\rho}.$$

Thus the scalar-model and radial-rescaling error is at most

$$\frac{(3K+1)t}{\rho}. \tag{32}$$

Combining (29), (30), (31), and (32), we obtain

$$|\Phi_t(f) - \langle v|f(B)|u\rangle| \le \frac{Km\rho s_0^2}{2t} + 2Ks_0 + \frac{4Kmt}{\rho g} \log \frac{1}{s_0} + \frac{(3K+1)t}{\rho}.$$

Take $s_0 = 4t/(\rho g)$. If $t \le g/16$, then $s_0 \le 1/2$ and (26) holds. Using $g \le 2$, $m \ge 1$, and $\rho \ge 1/2$, all terms are bounded by the right-hand side of (28) after increasing a universal constant C_0. □

Combining Theorems 4.2 and 4.4, choose, for example,

$$0 < t \le \min\left\{ \frac{g}{16}, \frac{c_1 \varepsilon_0 g^2}{(K+1)m\left[1+\log\left(\frac{16(K+1)m}{\varepsilon_0 g}\right)\right]} \right\}, \tag{33}$$

where c_1 is a sufficiently small universal constant. Then

$$|\Phi_t(f)| \ge \frac{\varepsilon_0}{2}. \tag{34}$$

By (19), if $K, d = \mathrm{poly}(n)$, such a t is at least inverse polynomial.

4.4 Embedding the Floquet closure into a log-local Hamiltonian

Let $U = G_{m-1} \cdots G_1$ and insert an identity gate so that $G_1 = I$. Use the Jacobi coefficients from B in the scaled open history Hamiltonian

$$H_{\mathrm{open}}(s) = \rho s \sum_{j=1}^{m} a_j |j\rangle\langle j| \otimes I \tag{35}$$

$$+ \rho s \sum_{j=1}^{m-1} b_j \left(|j+1\rangle\langle j| \otimes G_j + |j\rangle\langle j+1| \otimes G_j^{\dagger} \right). \tag{36}$$

Add the closure

$$H_{\mathrm{close}}(t) = t a_1 \left(|1\rangle\langle m| + |m\rangle\langle 1| \right) \otimes I \tag{37}$$

$$+ t b_1 \left(|2\rangle\langle m| + |m\rangle\langle 2| \right) \otimes I. \tag{38}$$

Set $H_{s,t}(U) = H_{\mathrm{open}}(s) + H_{\mathrm{close}}(t)$. Under the history gauge W, the open part becomes $\rho s B \otimes I$. Since $V_1 = V_2 = I$ and $V_m = U$, the two closure edges become

$$t\left(C \otimes U + C^{\dagger} \otimes U^{\dagger} \right).$$

Thus, on an eigenvector of U with eigenvalue $\mathrm{e}^{i\theta}$, the clock fiber is exactly $J_{s,t}(\theta)$. Consequently,

$$\mathrm{Tr}\, f(H_{s,t}(U)) = \sum_{\mathrm{e}^{i\theta} \in \mathrm{spec}(U)} \mathrm{Tr}\, f(J_{s,t}(\theta)). \tag{39}$$

The Hamiltonian has dimension mN and is $O(\log m + \log n)$-local. Its norm is at most one.

4.5 Fourier extraction and the Lipschitz hardness theorem

For $U_\phi = \mathrm{e}^{i\phi} U$, define

$$Y(s, \phi) = \frac{1}{mN} \mathrm{Tr}\, f(H_{s,t}(U_\phi)). \tag{40}$$

Using (39) and changing variables in the phase integral,

$$Z := \frac{1}{2\pi t} \int_0^1 \int_0^{2\pi} \mathrm{e}^{-i\phi} Y(s,\phi)\, \mathrm{d}\phi\, \mathrm{d}s = \frac{\Phi_t(f)}{m} \tau(U). \tag{41}$$

Because $Y(s, \phi)$ is real, its cosine and sine moments separately give the real and imaginary parts. In particular, the cosine moment determines $\mathrm{Re}\, \tau(U)$.

The maps $s \mapsto Y(s, \phi)$ and $\phi \mapsto Y(s, \phi)$ are Lipschitz with constants at most K and Kt, respectively. Therefore the two integrals in (41) can be approximated on polynomial-size grids when K and $1/t$ are polynomial. If every normalized trace is estimated to error η_{tr}, the induced error in Z is at most η_{tr}/t, up to the chosen inverse-polynomial quadrature error. Dividing by $|\Phi_t(f)|/m$ gives

$$|\widetilde{\tau}(U) - \tau(U)| \le O\left(\frac{m}{\varepsilon_0 t} \eta_{\mathrm{tr}} \right) + \text{quadrature error}. \tag{42}$$

Theorem 4.5 (Lipschitz hardness without $E_d/E_{d-1} < 1/2$). *Let $f_n : [-1,1] \to [-1,1]$ be a family with*

$$K_n := \mathrm{Lip}(f_n) = \mathrm{poly}(n), \qquad d_n := \widetilde{\deg}_{\varepsilon_0}(f_n) = \mathrm{poly}(n)$$

for a fixed $\varepsilon_0 > 0$. Assume that the alternation data and the Jacobi coefficients in Theorem 4.2 are constructible to polynomial precision, and that the clock length $m = d_n + 1$ is sufficient to contain the source DQC1 circuit (equivalently, the source circuit length is at most $d_n - O(1)$). Then normalized trace estimation

$$\frac{\mathrm{Tr}\, f_n(A)}{\dim A}$$

for log-local Hermitian A with $\|A\| \le 1$ is DQC1-hard at a sufficiently small inverse-polynomial additive accuracy, under polynomial-time classical Turing reductions. No condition on E_{d_n}/E_{d_n-1} is required.

More quantitatively, choose t as in (33). Estimating each normalized trace to

$$\eta_{\mathrm{tr}} \le c_2 \frac{\varepsilon_0 t}{m} \delta_U \tag{43}$$

allows one to estimate $\tau(U)$ to additive error δ_U, where c_2 is a sufficiently small universal constant. If $K_n, d_n, 1/\delta_U$ are polynomially bounded, then both $1/t$ and $1/\eta_{\mathrm{tr}}$ are polynomially bounded.

Proof. The open Jacobi witness follows from Theorem 4.2. The eigenvalue gap is bounded by (19). Choose t according to (33); then Theorem 4.4 gives (34). Construct the log-local Hamiltonians $H_{s,t}(U_\phi)$ and use (41). Polynomial-grid quadrature and normalized-trace estimates recover Z. Finally divide by $\Phi_t(f)/m$, using $|\Phi_t(f)| \ge \varepsilon_0/2$. The error condition is (43). DQC1-hardness follows from the DQC1-completeness of normalized unitary trace estimation at inverse-polynomial promise gap. □

Corollary 4.6 (DQC1-completeness under the membership hypothesis). *Under the hypotheses of Theorem 4.5, normalized trace estimation is DQC1-complete at inverse-polynomial accuracy, because the phase-estimation algorithm for polynomially Lipschitz functions places the problem in DQC1 [4, Proposition 2.3].*

5 Comparison of the two routes

	Uniform Chebyshev clock	Lipschitz f-dependent Floquet clock
Clock parameters	$a_j = 0$, $b_j = 1/2$; only the length r depends on f	All Jacobi parameters may depend on the alternation witness of f
Function hypothesis	A polynomially bounded D with small E_D	$K = \mathrm{Lip}(f) = \mathrm{poly}(n)$
Quantitative condition	$(D - d + 1)(E_D + \eta_{\mathrm{tr}}) \lesssim \varepsilon_0 \delta_U$	$\eta_{\mathrm{tr}} \lesssim \varepsilon_0 t \delta_U / m$, with $t = 1/\mathrm{poly}(n)$
Use of approximate degree	Forces a non-negligible Chebyshev coefficient in degrees $d, \ldots, D$	Produces a constant-strength endpoint dual witness
Use of Lipschitz regularity	Not needed	Separates alternation nodes and controls finite-t perturbation
$E_d/E_{d-1} < 1/2$	Not used	Not used
Reduction type in this note	Polynomial-time truth-table/Turing reduction over phases	Polynomial-time Turing reduction over scale and phase

6 Remarks, limitations, and open directions

(1) The first theorem does not establish an unconditional statement for every continuous function: polynomially fast vanishing of E_D is an extra hypothesis.

(2) The Lipschitz theorem avoids all higher-degree approximation errors. Its central new object is the scale-averaged coefficient (22). The factor $1/t$ does not create a superpolynomial cost because (19) permits an inverse-polynomial choice of t.

(3) The closed fiber in the Lipschitz route has two finite-rank closure terms whenever $a_1 \neq 0$. Thus it is a Floquet clock rather than a strict scalar periodic Jacobi matrix. A natural open problem is to realize the same endpoint functional with a strict periodic Jacobi operator.

(4) The theorem is stated under polynomial-time Turing reductions. A selector-register construction can package polynomially many rational quadrature points into a block-diagonal log-local Hamiltonian. Turning the signed Fourier weights into a single standard promise instance requires bookkeeping of a known offset and is not needed for the present conclusion.

(5) For a uniform complexity theorem, one needs an effective representation of the function family f_n and polynomial-bit construction of the alternation points. Remez-type algorithms and the tridiagonal inverse-spectral construction are discussed in [7, 4].

A Proof of the perturbation estimate

For completeness, we recall a standard argument behind Theorem 4.3; see Kato [5] or Bhatia [1]. Let $Aq_j = \lambda_j q_j$ and let $P_j = |q_j\rangle\langle q_j|$, $Q_j = I - P_j$. For $\|E\| \leq \gamma/4$, the eigenvalue near λ_j is isolated. Write its normalized eigenvector as $\psi = aq_j + \chi$, $\chi \in \operatorname{ran} Q_j$. Projecting $(A+E)\psi = \mu\psi$ onto $\operatorname{ran} Q_j$ and using

$$\left\|(Q_j(A-\mu)Q_j)^{-1}\right\| \le \frac{2}{\gamma}$$

gives $\|\chi\| \le 2\|E\|/\gamma$. Projecting onto q_j yields

$$\mu - \lambda_j - \langle q_j|E|q_j\rangle = \frac{\langle q_j|E|\chi\rangle}{a} + \left(\frac{1}{a} - 1\right)\langle q_j|E|q_j\rangle.$$

Since $|a| \ge \sqrt{1 - 4\|E\|^2/\gamma^2} \ge \sqrt{3}/2$, the right-hand side is bounded by $4\|E\|^2/\gamma$ after a harmless relaxation of constants.

B Quadrature cost

Let

$$F(s,\phi) = \frac{1}{mN}\operatorname{Tr} f(H_{s,t}(U_\phi)).$$

For Hermitian A, B of the same dimension, Weyl's inequality gives

$$\left|\frac{1}{L}\operatorname{Tr} f(A) - \frac{1}{L}\operatorname{Tr} f(B)\right| \le K\|A - B\|.$$

Since $\left\|H_{s,t} - H_{s',t}\right\| \le |s - s'|$ and $\left\|H_{s,t}(U_\phi) - H_{s,t}(U_{\phi'})\right\| \le t|\phi - \phi'|$, we have

$$|F(s,\phi) - F(s',\phi')| \le K|s - s'| + Kt|\phi - \phi'|.$$

After the prefactor $1/t$ in (41), rectangular quadrature with mesh widths inverse polynomial in $K, 1/t, m, 1/\delta_U$ gives the required inverse-polynomial accuracy and uses only polynomially many normalized-trace instances.

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