Quantum and Classical Query Complexities of Functions of Matrices

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Outline of the talk

- Background and motivations
- Lower bounds of query complexity
- BQP-completeness
- Conclusions

Quantum linear algebra

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  - Quantum recommendation systems: computing $A_{\geq\delta}|i\rangle$, where $A_{\geq\delta}$ is the truncation by keeping singular values larger than δ [Kerenidis-Prakash, '16].
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  - Solving linear differential equations: computing $e^{At}|\mathbf{b}\rangle$ [Berry, '10].
- All these problems can be described as functions of matrices: computing $f(A)|\mathbf{b}\rangle$.
- Can be solved by a similar idea to HHL, but more efficiently by quantum singular value transform [Gilyén-Su-Low-Wiebe, '18].

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Still works if f is not a polynomial, just consider its polynomial approximation.

This talk

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To better understand the above questions, we consider the following weaker problem

Problem (Approximate an entry of $f(A)$)

Let $f(x) : [-1,1] \to [-1,1]$ be a function, let A be sparse and Hermitian with $\|A\| \le 1$. Given two indices i, j and accuracy ε, compute $\langle i|f(A)|j\rangle \pm \varepsilon$.

If $A = UDU^{-1}$ is the eigenvalue decomposition, then $f(A) := Uf(D)U^{-1}$, where f is applied to the eigenvalues.

A is called s-sparse if the number of nonzero entries in each row/column is at most s.

Query complexity

For a sparse matrix $A = (A_{i,j})$, we are given 2 (commonly used) oracles:

$$(i,j) \longrightarrow \boxed{\mathcal{O}_1} \longrightarrow p_{i,j}$$

$$(i,j) \longrightarrow \boxed{\mathcal{O}_2} \longrightarrow A_{i,j}$$

where p_{ij} is the index of the j-th nonzero entry in the i-th row.

In the quantum case, we assume the oracles can be used in superposition, e.g., $\sum \alpha_{i,j}|i,j\rangle|0\rangle \mapsto \sum \alpha_{i,j}|i,j\rangle|A_{i,j}\rangle$

The query complexity is the minimal number of calls to the oracles to solve the problem.

An example: matrix powers A^m

Classical algorithms:

- Assume A is s-sparse, then by definition

$$(A^m)_{i,j} = \sum_{k_1} \sum_{k_2} \cdots \sum_{k_{d-1}} A_{i,k_1} A_{k_1,k_2} \cdots A_{k_{m-1},j}$$

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Quantum algorithms:

- Upper bound by QSVT $O(s\sqrt{m}/\varepsilon)$
- Our result (lower bound): $\Omega(\sqrt{m})$

General case (our results)

Assume f is continuous, A is s-sparse and Hermitian, then computing $f(A)_{i,j} \pm \varepsilon$ costs

	Quantum algorithm	Classical algorithm
Upper bound	$O(sd/\varepsilon)$	$O(s^{d-1})$
Lower bound	$\Omega(d)$	$\widetilde{\Omega}((s/2)^{(d-1)/6})$

where $d = \widetilde{\deg}_\varepsilon(f)$ is the approximate degree:

$$\begin{aligned}\widetilde{\deg}_\varepsilon(f) \quad = \quad & \min\{d : |f(x) - g(x)| \leq \varepsilon, \forall x \in [-1,1], \\ & \qquad g(x) \text{ is a polynomial of degree } d\}.\end{aligned}$$

The quantum lower bound is similar to the famous polynomial method for Boolean functions [Beals, Buhrman, Cleve, Mosca, de Wolf, FOCS '98].

Key theorem in the proofs

Theorem (Key theorem)

Let $f : [-1,1] \to [-1,1]$ *be continuous with odd part* $f_{\mathrm{odd}}(x) = \frac{f(x)-f(-x)}{2}$, *then there is a symmetric tridiagonal matrix*

$$A = \begin{pmatrix} 0 & b_1 & & & \\ b_1 & 0 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & 0 \end{pmatrix}_{n\times n}$$

satisfying $b_i \neq 0$, $\|A\| \leq 1$ *and* $f(A)_{1,n} \geq \varepsilon$, *where* $n = \widetilde{\deg}_{\varepsilon}(f_{\mathrm{odd}}) + O(1)$.

Proof. linear semi-infinite programming + dual polynomial method + properties of tridiagonal matrices. ■

Lower bound's proof of quantum algorithms

Parity problem: Given $x_1, \ldots, x_n \in \{0,1\}$, compute $x_1 \oplus \cdots \oplus x_n$, the quantum query complexity is $\Theta(n)$

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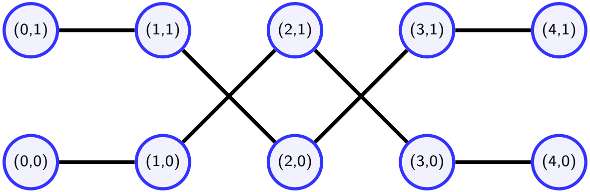
We construct a weighted graph G:

- **vertices:** (i, b), where $i \in \{0, 1, \ldots, n\}, b \in \{0, 1\}$
- **edges:** an edge between $(i-1, b)$ and $(i, b \oplus x_i)$
- **weights:** chosen carefully

For example, $(x_1, x_2, x_3, x_4) = (0, 1, 1, 0)$, then G is



Lower bound's proof of quantum algorithms

Essentially, G consists of two paths

$$(0,0) - (1, x_1) - (2, x_1 \oplus x_2) - \cdots - (n, x_1 \oplus \cdots \oplus x_n)$$
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Let A be the adjacency matrix of G (essentially two symmetric tridiagonal matrices).

- **Case 1:** if $x_1 \oplus x_2 \oplus \cdots \oplus x_n = 0$, then $\langle 0,0|f(A)|n,1\rangle = 0$

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As a result, if we can estimate $\langle 0,0|f(A)|n,1\rangle$, we then can solve the parity problem, which is known hard. It is very important to ensure that $n = \widetilde{\deg}(f) + O(1)$ in the key theorem. This idea is inspired by the no fast-forwarding theorem. [Berry, Ahokas, Cleve, Sanders, Comm. Math. Phys. '07]

Lower bound's proof of classical algorithms

Forrelation problem (Aaronson & Ambainis, 2015):

Given $g_1, g_2 : \{0,1\}^n \to \{\pm 1\}$, let $D_i = \mathrm{diag}(g_i(x) : x \in \{0,1\}^n)$, $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, define

$$\begin{aligned} \Phi(g_1, g_2) &:= \langle 0^n | H^{\otimes n} D_1 H^{\otimes n} D_2 H^{\otimes n} | 0^n \rangle \\ &= \frac{1}{2^{3n/2}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} g_1(x) g_2(y). \end{aligned}$$

The goal is to compute $\Phi(g_1, g_2) \pm 1/3$

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For this problem, the classical query complexity is lower bounded by $\Omega(\sqrt{2^n}/n)$, while the quantum query complexity is 1.

Feynman's clock construction

Let $U = U_{N-1} \cdots U_2 U_1$ be a unitary operator, define

$$A = \begin{pmatrix} 0 & b_1 U_1^\dagger & & & \\ b_1 U_1 & 0 & b_2 U_2^\dagger & & \\ & b_2 U_2 & \ddots & \ddots & \\ & & \ddots & \ddots & \end{pmatrix}$$

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Let $|\psi_t\rangle := |t\rangle \otimes U_t \cdots U_1 |0\rangle$, then

$$A|\psi_t\rangle = b_{t-1}|\psi_{t-1}\rangle + b_{t+1}|\psi_{t+1}\rangle$$

In subspace $\{|\psi_t\rangle : t = 0, 1, \dots, N-1\}$, A is a symmetric tridiagonal matrix of dimension N.

Lower bound's proof of classical algorithms

In the Forrelation problem, we have $U = H^{\otimes n} D_1 H^{\otimes n} D_2 H^{\otimes n}$. To ensure A is sparse in the clock construction, we decompose

$$H^{\otimes n} = (H \otimes I \otimes \cdots \otimes I)(I \otimes H \otimes \cdots \otimes I) \cdots (I \otimes I \otimes \cdots \otimes H)$$

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Now $N = 3n + 2$,

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BQP-completeness

Definition (PromiseBQP)

PromiseBQP is the set of promise problems $(\Pi_{\text{YES}}, \Pi_{\text{NO}})$ *that can be solved by a uniform family of quantum circuits. Namely, there is a quantum circuit* Y *such that the application of* Y *to the computational basis state* $|x, 0\rangle$ *produces the state*

$$Y|x, 0\rangle = \alpha_{x,0}|0\rangle \otimes |\psi_{x,0}\rangle + \alpha_{x,1}|1\rangle \otimes |\psi_{x,1}\rangle$$

such that

- *for every* $x \in \Pi_{\text{YES}}$ *it holds that* $|\alpha_{x,1}|^2 \geq 2/3$ *and*
- *for every* $x \in \Pi_{\text{NO}}$ *it holds that* $|\alpha_{x,1}|^2 \leq 1/3$*.*

Equivalently, $|\alpha_{x,1}|^2 - |\alpha_{x,0}|^2 \geq 1/3$ if $x \in \Pi_{\text{YES}}$ and $|\alpha_{x,1}|^2 - |\alpha_{x,0}|^2 \leq -1/3$ if $x \in \Pi_{\text{NO}}$.

Why we care about BQP-completeness

- It defines the hardest problems in BQP
  BQP = problems can be solved efficiently on a quantum computer

- It characterizes the power of quantum computing
  So we learn what quantum computers can do that classical computers may struggle with.

  If a BQP-complete is outside P, it confirms that quantum computers can solve problems that classical ones cannot efficiently.

Known examples: computing e^{iAt}, A^{-1}, A^{d}; approximating the Jones polynomial, etc.

Entry estimation problem

Problem

Let $f(x) : [-1, 1] \to [-1, 1]$ be a continuous function, let A be an $n \times n$ sparse Hermitian matrix such that $\|A\| \leq 1$. Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ and i, j be two indices. Assume that one of the following holds:

- *YES case: if $f(A)_{ij} \geq \varepsilon_1$,*
- *NO case: if $f(A)_{ij} \leq -\varepsilon_2$.*

Decide which is the case.

Entry estimation problem

Problem

Let $f(x) : [-1, 1] \to [-1, 1]$ be a continuous function, let A be an $n \times n$ sparse Hermitian matrix such that $\|A\| \leq 1$. Let $\varepsilon_1, \varepsilon_2 \in (0, 1)$ and i, j be two indices. Assume that one of the following holds:

- *YES case: if $f(A)_{ij} \geq \varepsilon_1$,*
- *NO case: if $f(A)_{ij} \leq -\varepsilon_2$.*

Decide which is the case.

Theorem

Assume that $\widetilde{\deg}(f) = \Omega(\mathrm{polylog}(n))$. Then the "entry estimation problem" is PromiseBQP-complete.

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- Let $U = Y^\dagger Z_1 Y$, then [replace $|0\rangle$ with $|x,0\rangle$]

$$\begin{aligned}
\langle \phi_{N-1}|f(A)|\psi_0\rangle &= \langle \psi_{N-1}|f(A)|\psi_0\rangle \cdot \langle x,0|U|x,0\rangle \\
&= \langle \psi_{N-1}|f(A)|\psi_0\rangle \cdot (|\alpha_{x,1}|^2 - |\alpha_{x,0}|^2) \\
&= \begin{cases} \geq \varepsilon/3 & \text{if } x \in \text{YES} \\ \leq -1/3 & \text{if } x \in \text{NO} \end{cases}
\end{aligned}$$

because $\langle \psi_{N-1}|f(A)|\psi_0\rangle \in [\varepsilon, 1]$.

Background and motivations

Lower bounds of query complexity

BQP-completeness

Conclusion

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- For functions of matrices, we proved

	Quantum algorithm	Classical algorithm
Upper bound	$O(sd/\varepsilon)$	$O(s^{d-1})$
Lower bound	$\Omega(d)$	$\Omega((s/2)^{(d-1)/6})$

- From the point of approximate degree,
  - quantum algorithm is optimal
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Open questions:

- Lower bound in terms of ε in the quantum case? $\Omega(1/\varepsilon)$?
- Any applications to quantum PCP? [Gharibian, Le Gall, Dequantizing the Quantum Singular Value Transformation, STOC'22]

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Thanks very much for your time!