Randomized quantum singular value transformation

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based on joint work with Shantanav Chakraborty, Soumyabrata Hazra,
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(arXiv:2504.02385, 91 pages)

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- It applies polynomial transformations on the singular values of an operator A, provided A is embedded in the top-left block of a unitary, known as block encoding. Namely, (I will assume A is Hermitian and $\|A\| \leq 1$ in this talk)

$$\underbrace{U := \begin{bmatrix} A & * \\ * & * \end{bmatrix}}_{\text{(unitary)}} \xrightarrow[\text{QSVT}]{\text{poly } f(x)} \underbrace{\widetilde{U} = \begin{bmatrix} f(A) & * \\ * & * \end{bmatrix}}_{\text{(unitary)}}$$

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- Usually, the most technical part is constructing U efficiently.

Previous works on QSVT

- The quantum circuit for QSVT: Assume $f(x) \in \mathbb{C}[x]$, degree d, even/odd, and $|f(x)| \le 1$ for all $x \in [-1, 1]$, then there exists $\Phi := (\phi_1, \ldots, \phi_d) \in \mathbb{R}^d$, s.t.

$$\begin{bmatrix} f(A) & * \\ * & * \end{bmatrix} = U_\Phi = \begin{cases} e^{\mathbf{i}\phi_1 Z} U \prod_{j=1}^{(d-1)/2} \left(e^{\mathbf{i}\phi_{2j} Z} U^\dagger e^{\mathbf{i}\phi_{2j+1} Z} U \right), & \text{if } d \text{ is odd,} \\ \prod_{j=1}^{d/2} \left(e^{\mathbf{i}\phi_{2j-1} Z} U^\dagger e^{\mathbf{i}\phi_{2j} Z} U \right), & \text{if } d \text{ is even.} \end{cases}$$

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- Assuming access to $U = e^{\mathbf{i}H}$, then there exists Φ such that

$$U_\Phi = \begin{bmatrix} P(\cos(A)) & * \\ * & * \end{bmatrix}, \quad H = \begin{bmatrix} & A^\dagger \\ A & \end{bmatrix}, \quad f(x) = P(\cos(x)).$$

See [Lloyd et al, arXiv:2104.01410; Dong-Lin-Tong, PRX Quantum 2022]

Motivations

- Assume $A = \sum_{k=1}^{L} \lambda_k P_k$ is a unitary (e.g., Pauli) decomposition. Then how to do QSVT?

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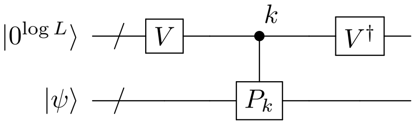
- Linear combination of unitaries (LCU): By LCU, we can construct a block encoding of A. Circuit depth is $O(L)$, number of ancilla qubits is $O(\log L)$.



Without loss of generality, we assume $\lambda_k > 0$ and $\sum_k \lambda_k = 1$, then

$$V|0\rangle = \sum_{k=1}^{L} \sqrt{\lambda_k}\,|k\rangle.$$

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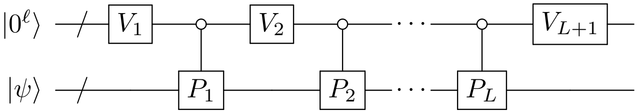

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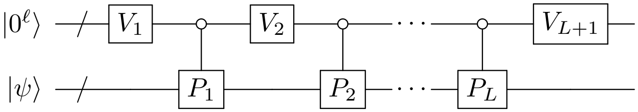

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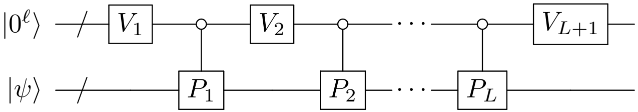

- Rmk. The above lower bound might give some hint about the number of ancillas needed for multivariate QSP/QSVT theory!?
- Question: Can we construct an approximate block-encoding of A using 1 or $O(1)$ ancilla qubits? [Gilyén-Vasconcelos obtained such a result for multiplication of block encodings]

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- Prior work: qDRIFT.

  If $f(x) = e^{\mathbf{i}xt}$, then we can use qDRIFT [Campbell, PRL'19]. Define distribution $\mathcal{D} = \{(\lambda_k, P_k)\}_{k=1}^{L}$. Sample ℓ Paulis $P_{j_1}, \ldots, P_{j_\ell}$, then on average (for some β)

$$e^{\mathbf{i}\beta P_{j_1}} e^{\mathbf{i}\beta P_{j_2}} \cdots e^{\mathbf{i}\beta P_{j_\ell}} \approx e^{\mathbf{i}Ht}$$

  - qDRIFT uses 0 ancilla qubit,
  - $\ell \approx t^2$ is independent of L.
  - It shows advantages over Trotter/QSVT when $t \ll L$.

Our results

Theorem 2

Let $f(x) \in \mathbb{C}[x]$ be a degree-d polynomial such that $|f(x)| \le 1$ for all $x \in [-1, 1]$ and has parity-($d \mod 2$). Assume A is Hermitian with $\|A\| \le 1$ and

$$U = \begin{bmatrix} cI & sA \\ sA^\dagger & -cI \end{bmatrix},$$

where

$$n = \widetilde{\Theta}(d), \quad s = 1/\sqrt{n}, \quad c = \sqrt{1 - 1/n}, \quad \alpha = \widetilde{\Theta}(\sqrt{n}).$$

Then there exists $\Phi \in \mathbb{R}^n$ such that

$$U_\Phi \approx \begin{bmatrix} * & f(A/\alpha) \\ * & * \end{bmatrix} \text{ if } d \text{ is odd, and } U_\Phi \approx \begin{bmatrix} * & * \\ * & f(A/\alpha) \end{bmatrix} \text{ if } d \text{ is even.}$$

Rmk. U and U_Φ may not be unitary. U is unitary iff A is unitary. The above is comparable to the key theorem for QSVT.

Randomized QSVT

- Recall (assume n is even for convenience)

$$U = \begin{bmatrix} cI & sA \\ sA^{\dagger} & -cI \end{bmatrix}, \quad U_{\Phi} = \prod_{j=1}^{n/2} \left(e^{\mathbf{i}\phi_{2j-1} Z} {\color{red}U^{\dagger}} e^{\mathbf{i}\phi_{2j} Z} {\color{red}U} \right).$$

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- For $A = \sum_{k=1}^{L} \lambda_k P_k$, we consider the distribution $\mathcal{D} = \{(\lambda_k, P_k)\}_{k=1}^{L}$. We independently sample n Paulis $P_{j_1}, \ldots, P_{j_n}$ and replace each U or $U^\dagger$ with

$$U_{j_k} = \begin{bmatrix} cI & sP_{j_k} \\ sP_{j_k}^\dagger & -cI \end{bmatrix}$$

Denote the resulting unitary as $U_\Phi^{(J)}$ with $J = (j_1, \ldots, j_n)$. Then

$$\mathbb{E}_J[U_\Phi^{(J)}] = U_\Phi.$$

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- This has also been observed in several previous randomized quantum algorithms: qDRIFT, qSWIFT, and randomized LCU.

  [Nakaji et al., PRX Quantum'24; Wang et al. PRX Quantum'24; Chakraborty, Quantum'24]

Lower bounds

Theorem 3 (Sample complexity)

Given $\mathcal{D} = \{(\lambda_k, P_k)\}_{k=1}^{L}$, $\Omega(t^2/\varepsilon^2)$ *samples are required to estimate* $\langle \psi_1 | e^{\mathbf{i}At} | \psi_0 \rangle \pm \varepsilon$ *for any randimized quantum algorithms with only access to* $\mathcal{D}$.

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A randomized quantum algorithm has the following form:

$$W_{Q_1,\ldots,Q_c} = U_1 Q_1 U_2 Q_2 \cdots U_c Q_c U_{c+1},$$

- $Q_1, \ldots, Q_c$ are randomly and independently generated unitaries, each depends on a random unitary from $\{P_1, \ldots, P_L\}$.
- $U_1, \ldots, U_{c+1}$ are unitaries independent of $\{P_1, \ldots, P_L\}$.
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Question: Can we prove a lower bound of $\Omega(d^2)$ for general polynomials?

Other results

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