Quantum communication complexity of linear regression

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based on joint work with Ashley Montanaro
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Background

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A fundamental problem:

Solving linear system. **Given some quantum access to a matrix $A \in \mathbb{R}^{m \times n}$, and the ability to prepare $\lvert \mathrm{b} \rangle$, output $\lvert A^{-1}\mathrm{b} \rangle$.**

Here A^{-1} means the pseudoinverse if it is not invertible.

Quantum algorithms for linear systems

Theorem (Harrow, Hassidim, Lloyd 2008 + many others[1])

There is a quantum algorithm that solves this problem in cost

$$\kappa \cdot \mathrm{poly}(\log(mn), \log 1/\varepsilon),$$

where κ is the condition number.

Leads to wide applications, especially in machine learning, e.g., recommendation systems [Kerenidis and Prakash, 1603.08675].

[1] [Ambainis 1010.4458], [Childs, Kothari, Somma 1511.02306], [Wossnig, Zhao, Prakash 1704.06174], [Chakraborty, Gilyén, Jeffery 1804.01973], ...

Quantum-inspired classical algorithms

In 2018, Tang showed that assuming a similar data structure to QRAM, there is a classical algorithm that can solve the recommendation systems in time polylog in the dimension. [Tang, 1807.04271].

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$\Rightarrow$ no quantum exponential speedups for many machine learning problems in the low-rank case. [Chia et al, 1910.06151].

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- Query complexity
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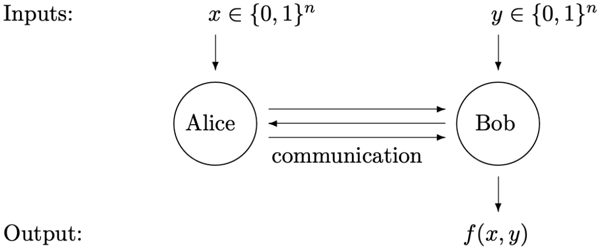
Theorem (Our result, informal)

Quantum computers can have exponential speedups in terms of communication complexity for some fundamental linear algebra problems.

Communication complexity: classical case

Let f be a function, say from $\{0,1\}^n \times \{0,1\}^n$ to $\{0,1\}$. Alice receives $x \in \{0,1\}^n$ and Bob receives $y \in \{0,1\}^n$, they want to compute $f(x,y)$ together.[1]



The communication complexity is counted by the number of bits used in the communication.

[1] Ronald de Wolf, Quantum Computing: Lecture Notes, 1907.09415.

Communication complexity: classical case

Local costs are not considered in communication complexity, and we usually assume Alice and Bob have unlimited computational powers.

The communication can be 1-way or 2-way:

1. 1-way: Only Alice can send information to Bob, or the other way around.
2. 2-way: Alice and Bob can send information to each other.

Communication complexity: quantum case

In the quantum case, Alice and Bob can send quantum states to each other. The complexity is counted by the number of qubits used.

Solving linear regressions: 2-party case

Problem statement.

Alice: $A \in \mathbb{R}^{m \times n}$
Bob: $\mathbf{b} \in \mathbb{R}^{m}$
Goal: Solve $\arg\min \|A\mathbf{x} - \mathbf{b}\|$

More precisely,

- **in the quantum case: output** $|A^{-1}\mathbf{b}\rangle$ **up to error** ε
- **in the classical case: output samples from a probability distribution** ε**-close to the one defined by** $|A^{-1}\mathbf{b}\rangle$

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For simplicity, we assume that the entries of $A, \mathbf{b}$ are specified by $\mathrm{polylog}(mn)$ bits.

Our results

	Alice $\to$ Bob	Bob $\to$ Alice	Alice $\leftrightarrow$ Bob
Q	$\widetilde{O}(\kappa^2 \min(m,n))$	$\widetilde{O}(\kappa^2)$	$\widetilde{O}(\kappa)$
	$\Omega(\min(m,n))$	$\Omega(\kappa^2)$	$\Omega(\kappa)$
C	$\widetilde{O}(mn)$	$\widetilde{O}(m)$	$\widetilde{O}(m)$
	$\Omega(\min(m,n))$	$\Omega(\min(m,n))$	$\Omega(\min(m,n))$
	at most quadratic	can be exponential	can be exponential

$\to$ 1-way communication; $\leftrightarrow$ 2-way communication.

$\kappa =$ condition number of A.

All lower bounds hold even if $m = n, \kappa = O(1)$.

Quantum protocol (eg): 1-way from Bob to Alice

- Bob sends the state $|0\rangle|\mathbf{b}\rangle$ to Alice.

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$$U = \begin{pmatrix} A^{-1}/\|A^{-1}\| & * \\ * & * \end{pmatrix}.$$

She then applies U to $|0\rangle|\mathbf{b}\rangle$:

$$\frac{1}{\|A^{-1}\|}|0\rangle \otimes A^{-1}|\mathbf{b}\rangle + |1\rangle|G\rangle.$$

The success probability is

$$\frac{\|A^{-1}|\mathbf{b}\rangle\|^2}{\|A^{-1}\|^2} \geq \frac{1}{\kappa^2}.$$

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- So Bob has to send $O(\kappa^2)$ copies of $|0\rangle|\mathbf{b}\rangle$ to Alice.

Lower bound analysis

We use the hardness of disjointness problem.

Recall: In this problem, Alice has $(x_1, \ldots, x_n) \in \{0, 1\}^n$ and Bob has $(y_1, \ldots, y_n) \in \{0, 1\}^n$. They want to determine if $x_i = y_i = 1$ for some i.

[2] Buhrman and de Wolf, Communication complexity lower bounds by polynomials, 2001

[3] Aaronson and Ambainis, Quantum search of spatial regions, 2003.

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Quantum: $\Theta(n)$ (1-way)[2], $\Theta(\sqrt{n})$ (2-way)[3]
Classical: $\Theta(n)$ (1-, 2-way)

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So the solution of $A\mathbf{x} = \mathbf{b}$ satisfies

$$(A^{-1}\mathbf{b})_i = \begin{cases} 1 & x_i = y_i = 1, \\ \varepsilon & x_i = 1 \text{ xor } y_i = 1, \\ \varepsilon^2 & x_i = y_i = 0. \end{cases}$$

Lower bound analysis

Let

$$P = \{i : x_i = y_i = 1\}, \quad Q = \{i : x_i = 1 \text{ xor } y_i = 1\},$$

then the quantum state of the solution is

$$\frac{1}{\|A^{-1}\mathbf{b}\|}\left(\sum_{i\in P}|i\rangle + \varepsilon\sum_{i\in Q}|i\rangle + \varepsilon^2\sum_{i\notin P\cup Q}|i\rangle\right).$$

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Choose $\varepsilon = 1/\sqrt{n}$, then

- If $|P| = 1$, the probability of seeing $i \in P$ is as large as a constant ($\approx 1/2$). $\rightarrow$ see the same index many times
- If $|P| = 0$, the state is close to a uniform superposition of indices from Q. $\rightarrow$ see different indices uniformly

Multi-party case

There are s players $P_1, \ldots, P_s$, each P_i has a matrix $A_i \in \mathbb{R}^{m_i \times n}$ and a vector $\mathbf{b}_i \in \mathbb{R}^{m_i}$, they want to solve

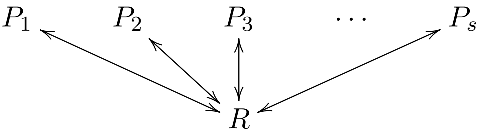
$$\arg\min_{\mathbf{x}} \quad \|A\mathbf{x} - \mathbf{b}\|,$$

where

$$A = \begin{pmatrix} A_1 \\ \vdots \\ A_s \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_s \end{pmatrix}.$$

The model we use

Similar to the classical coordinator model, we assume that the communication is 2-way between each player P_i and the referee R, we call this the quantum coordinator model.



Task: the referee outputs

- the quantum state $|A^{-1}\mathbf{b}\rangle \pm \varepsilon$ in the quantum case
- samples from a probability distribution ε-close to the one defined by $|A^{-1}\mathbf{b}\rangle$ in the classical case

The main result

	Upper bound	Lower bound
Quantum	$\widetilde{O}(s^{1.5}\kappa)$	$\Omega(s\kappa)$
Classical	$O(sn^2)$	$\Omega(sn)$

Exponential speedup exists when $s, \kappa \ll n$.

Main technique: Quantum singular value transformation (QSVT).[4]

[4] Gilyén, Su, Low, Wiebe, Quantum singular value transformation, STOC'19

QSVT

Suppose A is Hermitian and there is a unitary

$$U = \begin{pmatrix} A/\alpha & * \\ * & * \end{pmatrix}.$$

Let $f(x)$ be a bounded polynomial of degree d, then there is a quantum circuit that implements the unitary

$$\widetilde{U} = \begin{pmatrix} f(A/\alpha) & * \\ * & * \end{pmatrix}.$$

It uses $O(d)$ applications of $U, U^\dagger$ and some one- and two-qubit gates.

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$\Rightarrow$ So you can prepare the state $\propto f(A/\alpha)|\psi\rangle$.

QSVT

Theorem (QSVT w.r.t. communication complexity)

In the quantum coordinator model, there is a quantum protocol for the referee to use

$$\widetilde{U} = \begin{pmatrix} f(A/\alpha) & * \\ * & * \end{pmatrix}$$

with $O(sd \log n)$ qubits communication.

Application 1: linear regression

If $f(x) \approx 1/x$, then we have

Corollary (Linear regression)

There is a quantum protocol for linear regression in cost $\widetilde{O}(s^{1.5}\kappa)$.

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Recall the time complexity is $\widetilde{O}(T_A \alpha/\sigma_{\min})$,[5] where $\sigma_{\min}$ is the minimal nonzero singular value of A, T_A is the cost of constructing the block-encoding

$$U = \begin{pmatrix} A/\alpha & * \\ * & * \end{pmatrix}.$$

For communication complexity, we can compute $T_A = O(s \log n)$, $\alpha = O(\sqrt{s}\sigma_{\max})$ precisely.

[5] Gilyén, Su, Low, Wiebe, Quantum singular value transformation, STOC'19

Application 2: Hamiltonian simulation

Player P_i: has a Hamiltonian H_i of dimension n, $i \in [s]$

The referee R: has a quantum state $|\psi\rangle$

The task: prepare $e^{i(H_1+\cdots+H_s)t}|\psi\rangle$ up to error ε

Application 2: Hamiltonian simulation

If $f(x) \approx e^{ixt}$, then we have

Corollary (Hamiltonian simulation)

There is a quantum protocol with communication complexity

$$O\left((s \log n)\left(\sum_{i=1}^{s} \|H_i\| \, |t| + \frac{\log(1/\varepsilon)}{\log(e + (\sum_{i=1}^{s} \|H_i\| \, |t|)^{-1} \log(1/\varepsilon))}\right)\right).$$

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Recall the time complexity is[6]

$$O\left(T_H\left(\alpha|t| + \frac{\log(1/\varepsilon)}{\log(e+|\alpha t|^{-1}\log(1/\varepsilon))}\right)\right),$$

where $H = H_1 + \cdots + H_s$ and T_H is the complexity of constructing the block-encoding of H. For communication complexity, $T_H = O(s\log n), \alpha = O(\sum_{i=1}^{s}\|H_i\|)$.

[6] Gilyén, Su, Low, Wiebe, Quantum singular value transformation, STOC'19

Conclusions

QSVT can be dequantized $\rightarrow$ no exponential speedup in terms of time and query complexity for some problems.[7]

[7] Chia et al: 1910.06151; Jethwani, Le Gall, Singh: 1910.05699; Gharibian, Le Gall: 2111.09079

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QSVT can be dequantized → no exponential speedup in terms of time and query complexity for some problems.[7]

In quantum coordinator model, we can use QSVT → efficient quantum communication protocols.

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In quantum coordinator model, we can use QSVT $\rightarrow$ efficient quantum communication protocols.

$\Rightarrow$ For many problems where quantum computers lose exponential speedups in terms of time and query complexity, it is possible to have exponential speedups in terms of communication complexity.

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⇒ For many problems where quantum computers lose exponential speedups in terms of time and query complexity, it is possible to have exponential speedups in terms of communication complexity.

⇒ It is interesting to explore more, usually the hard part is the lower bound analysis.

[7] Chia et al: 1910.06151; Jethwani, Le Gall, Singh: 1910.05699; Gharibian, Le Gall: 2111.09079

Bristol is hiring postdocs to work on the theory of quantum computing. If you have any interest, please feel free to contact

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Thanks very much for your attention!