DQC1-completeness of Normalized Trace Estimation for Functions of Log-local Hamiltonians

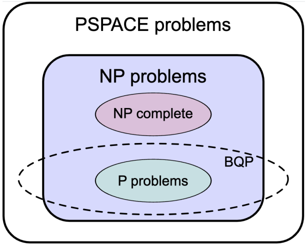
Changpeng Shao
(AMSS, CAS)

based on joint work with Zhengfeng Ji (Tsinghua U.), Tongyang Li (Peking U.)
Xinzhao Wang (Peking U.) and Yuxin Zhang (AMSS, CAS),
arXiv:2604.01519 (FOCS 2026)

2026.08.03-05
CQCC 2026 (Shenzhen)

Background

- When exploring quantum advantages, we are usually interested in BQP complete problems. Those problems demonstrate the full power of quantum computers.
  - Hamiltonian simulation e^{iAt}
  - Matrix inversion A^{-1}
  - Matrix powers A^d
  - Matrix exponential e^{At}



- Those problems can be summarized as computing $f(A)$ for some function $f(x)$.

Background

Problem to answer in this talk:

Problem (Normalized trace estimation)

Let $f : [-1, 1] \to [-1, 1]$ be continuous, A be a Hermitian matrix of dimension N, what is the complexity of estimating $\frac{1}{N}$ Tr$[f(A)]$?

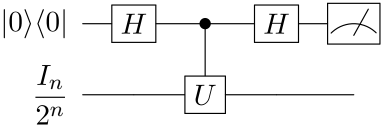
Main motivation: many problems reduce to such spectral sums, e.g.,

- Machine learning: log-det Tr$[\log A] = \log\det(A)$
- Statistical physics: partition function Tr$[e^{-\beta A}]$
- Graph theory: Tr$[A^d]$ counts closed walks
- Particularly, this problem is relevant to the DQC1 model, a weaker but simple quantum computational model than BQP.

Goal: when is this quantumly easy but classically hard?

The DQC1 (One clean qubit model) Model

- It is a simple quantum computing model introduced by Knill and Laflamme in 1998, motivated by the study of nuclear magnetic resonance quantum computers.



- The probability of measuring 0 is

$$\frac{1}{2} + \frac{1}{2}\frac{\mathrm{Re}(\mathrm{Tr}[U])}{2^n}$$

The real part can be estimated up to ε by repeating the circuit $O(1/\varepsilon^2)$ times.

The DQC1 (One clean qubit model) Model

- In DQC1 model, we can estimate $\frac{\mathrm{Tr}[U]}{2^n}$ efficiently.
- It is believed to be weaker than BQP model, but it can still solve some problems that are hard for classical computers.

Knill & Laflamme (1998)	Estimating $\frac{1}{2^n}\mathrm{Tr}(U)$ is DQC1-complete*
Shepherd (2006)	DQC1 = DQCk, $k = O(\log n)$
Shor & Jordan (2008)	Estimating Jones polynomial is DQC1-complete
Brandão (2008)	Estimating $\frac{1}{2^n}\mathrm{Tr}(e^{-\beta A})$ is DQC1-complete
Morimae, Fujii & Fitzsimons (2014)	Classically hard to simulate DQC1
Edenhofer, Hasegawa & Le Gall (2025)	Estimating $\frac{1}{2^n}\mathrm{Tr}(A^k)$, $\frac{1}{2^n}\mathrm{Tr}(A^{-1})$ is DQC1-complete
Shao & Wang (2025)	Estimating $\frac{1}{2^n}\mathrm{Tr}(\log A)$ is DQC1-complete

- The above results assume A is log-local.[†]

*DQC1-complete: problems capturing the full power of DQC1.

†log-local: $A = \sum_k A_k$ has size 2^n, each A_k acts nontivially on $\log(n)$ qubits.

Some reasons why people care about DQC1 model

- **Conceptual Simplicity and Implementability:**
  An advantage in designing experiments and analyzing outputs.
- **Minimal Qubit Purity Requirements:**
  Only 1 pure qubit.
  So DQC1 is "far less resource intensive than universal quantum computing".
- **Robustness Through Discord (Noise Resilience):**
  Does not rely on large-scale entanglement.
  Less sensitive to certain types of noise.
  Naturally suited to the imperfect, noisy quantum devices available today.
- **Computational Power on Specific Problems:**
  Solve specialized problems believed to be classically intractable.

Generalizations

- Many DQC1-complete problems can be written as

$$\frac{1}{2^n}\mathsf{Tr}[f(A)],$$

  where $f(x)$ is a continuous function.
- Approximate $f(x)$ by a polynomial

$$f(x) \approx g(x) = \sum_{k=0}^{d} a_k x^k.$$

- Then

$$\frac{1}{2^n}\mathsf{Tr}[f(A)] \approx \sum_{k=0}^{d} a_k \cdot \frac{1}{2^n}\mathsf{Tr}[A^k].$$

Generalizations

- For s-sparse A,

$$\frac{1}{2^n}\mathrm{Tr}[A^k] = \mathbb{E}[(A^k)_{ii}]$$

  can be estimated by sampling:
  - $O(1/\varepsilon^2)$ samples
  - each sample costs $O(s^k)$
- Total cost: $O(s^d/\varepsilon^2)$ [Edenhofer, Hasegawa, Le Gall, QIP 2026].
- **Efficient if:** $d = O(\log n)$
- So the complexity is essentially controlled by how well $f(x)$ can be approximated by a low-degree polynomial.

Approximate degree

- In complexity theory, we are interested in the approximate degree:

$$\widetilde{\deg}_\varepsilon(f) = \min\{d \in \mathbb{N} : |f(x) - g(x)| \leq \varepsilon \text{ for all } x \in [-1,1], \deg(g) = d\}.$$

- Appears in the lower bound analysis of many quantum/classical algorithms.
- With this notation,

$$\frac{1}{2^n}\mathrm{Tr}[f(A)]$$

  can be estimated in time $O(s^{\widetilde{\deg}_\varepsilon(f)}/\varepsilon^2)$ classically.
- Quantumly, it can be estimated in time $\widetilde{O}(s \cdot \widetilde{\deg}_\varepsilon(f))$. [Cade, Montanaro, TQC 2018]
- So it seems to have exponential quantum speedup in terms of $\widetilde{\deg}_\varepsilon(f)$.

Our results

Assume $f(x)$ is Lipschitz continuous from $[-1,1]$ to $[-1,1]$.

Result 1: Assume that A is s-sparse and Hermitian, then (assume $\varepsilon = \Omega(1)$)[‡]

	Quantum	Classical
Upper bound	$s \cdot \widetilde{\deg}_\varepsilon(f)$	$s^{\widetilde{\deg}_\varepsilon(f)}$
Lower bound	$\widetilde{\deg}_\varepsilon(f)$	$(s/2)^{\widetilde{\deg}_\varepsilon(f)/9}$

$\Rightarrow$ **This confirms the exponential speedup.**

Result 2: Assume that A is log-local and $\widetilde{\deg}_\varepsilon(f) = \Omega(\mathrm{poly}\log N)$, then estimating $\frac{1}{N}\mathrm{Tr}[f(A)]$ is DQC1-complete.

$\Rightarrow$ **This includes all previous partial results.**

[‡]Based on an idea of [Low, STOC 2018], the quantum complexity can be modified to $O(\sqrt{s}^{1+o(1)} \cdot \widetilde{\deg}_\varepsilon(f))$ and $\Omega(\sqrt{s} \cdot \widetilde{\deg}_\varepsilon(f))$.

Technical overview

Main tools:

- Periodic Jacobi matrix
- Chebyshev equioscillation theorem
- Feynman's circuit-to-Hamiltonian construction
- Trace version of the k-forrelation problem

Technical overview

- An n-qubit unitary $U = U_m \cdots U_2 U_1$, $m = \mathrm{poly}(n)$.
- Feynman's circuit-to-Hamiltonian construction: $(|m\rangle := |0\rangle)$

$$A = \sum_{t=1}^{m} a_t |t\rangle\langle t| \otimes I_n + \sum_{t=1}^{m} b_t \big(|t\rangle\langle t-1| \otimes U_t + |t-1\rangle\langle t| \otimes U_t^\dagger \big),$$

- A key observation: for any eigenpair $(e^{i\theta}, |\phi_\theta\rangle)$ of U, the restriction of A to the invariant subspace spanned by

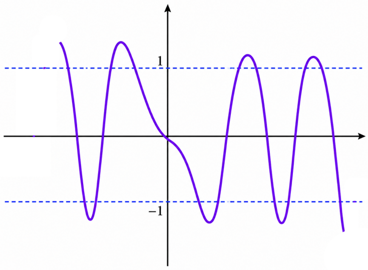
$$\Big\{ |0\rangle \otimes |\phi_\theta\rangle, \quad |1\rangle \otimes U_1 |\phi_\theta\rangle, \quad \ldots, \quad |m-1\rangle \otimes U_{m-1} \cdots U_2 U_1 |\phi_\theta\rangle \Big\}$$

is a periodic Jacobi matrix of the form

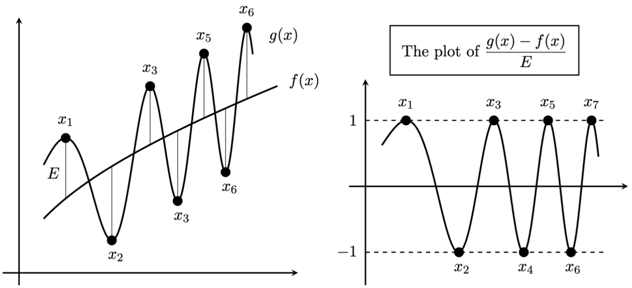
$$A_\theta := \begin{bmatrix} a_1 & b_1 & & & b_m e^{i\theta} \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{m-1} \\ b_m e^{-i\theta} & & & b_{m-1} & a_m \end{bmatrix}.$$

Technical overview

- Thus
$$\mathrm{Tr}[f(A)] = \sum_{\theta:\, e^{i\theta} \text{ is an eigenvalue of } U} \mathrm{Tr}[f(A_\theta)].$$
- Recall $\mathrm{Re}(\mathrm{Tr}[U]) = \sum_\theta \cos\theta$
- About periodic Jacobi matrix,
$$\det(xI - A_\theta) = 2b_1 \cdots b_m \big(\Delta(x) - \cos\theta\big),$$
where $\Delta(x)$ is independent of θ,
and $\mathrm{Tr}[\Delta(A_\theta)] = m\cos\theta$.
- A natural, but incorrect, idea: if $\Delta(x) \approx f(x)$, then $\mathrm{Tr}[f(A_\theta)] \approx m\cos\theta$.



Technical overview



- (crucial) Using Chebyshev equioscillation theorem to fix it.
- Roughly, $\Delta(x) \approx \dfrac{f(x) - g(x)}{E}$, from which we can reconstruct A.
- Then we obtain a connection between $\frac{1}{m2^n}\mathrm{Tr}[f(A)]$ and $\frac{1}{2^n}\mathrm{Re}(\mathrm{Tr}[U])$.
- $\Rightarrow$ Result 2: DQC1-completeness.

k-forrelation problem in DQC1 model

- Let $H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $\mathbf{x}_1, \ldots, \mathbf{x}_k \in \{1, -1\}^{2^n}$ with oracles $O_{\mathbf{x}_1}, \ldots, O_{\mathbf{x}_k}$ to query them.
- Let $U = O_{\mathbf{x}_1} H^{\otimes n} O_{\mathbf{x}_2} H^{\otimes n} \cdots H^{\otimes n} O_{\mathbf{x}_k} H^{\otimes n}$ and

$$\mathtt{Trace}_k(\mathbf{x}) := \frac{1}{2^n}\mathrm{Tr}[U].$$

- The classical query complexity of estimating $\mathtt{Trace}_4(\mathbf{x})$ is lower bounded by $\widetilde{\Omega}(2^{n/4})$. [Shepherd, PhD thesis, 2010].[§]
- Using the connection established above, one can prove a lower bound of

$$\widetilde{\Omega}((s/2)^{\widetilde{\deg}_\varepsilon(f)/9}).$$

[§] We conjecture that the lower bound is $\widetilde{\Omega}(2^{n(1-2/k)})$ for estimating $\mathtt{Trace}_k(\mathbf{x})$. So $\widetilde{\Omega}(2^{n/2})$ for estimating $\mathtt{Trace}_4(\mathbf{x})$.

Final remarks

- Our results were hold when $E_d/E_{d-1} < 1/2$, where $d = \widetilde{\deg}_\varepsilon(f)$ and

$$E_d := \min_{\deg(g) \leq d} \max_{-1 \leq x \leq 1} |f(x) - g(x)|$$

- Many functions satisfy this condition, and we believe this condition can be removed.

- Recently, with the help of GPT 5.6 Sol, we obtained two new results for DQC1 hardness:
  1. $E_D = o(1/D)$ for some $D = \mathrm{poly}(n) > d$. (still based on our current idea)
  2. Lipschitz condition. (established a new connection to [Montanaro, Shao, STOC 2024])

Thank you!