Faster quantum-inspired algorithms for solving linear systems

Changpeng Shao and Ashley Montanaro

School of Mathematics, University of Bristol, UK
5-8 July 2021, 16th TQC

arXiv:2103.10309

EPSRC Engineering and Physical Sciences Research Council

European Research Council Established by the European Commission

Background

Solving linear systems is a fundamental problem in machine learning

Background

Solving linear systems is a fundamental problem in machine learning

In the field of quantum computing, many algorithms are known, eg

- Harrow-Hassidim-Lloyd algorithm for sparse linear systems
- Block-encoding method for general linear systems

Background

Solving linear systems is a fundamental problem in machine learning

In the field of quantum computing, many algorithms are known, eg

- Harrow-Hassidim-Lloyd algorithm for sparse linear systems
- Block-encoding method for general linear systems

A useful way to construct block-encoding of a matrix is based on the QRAM data structure

Background

Solving linear systems is a fundamental problem in machine learning

In the field of quantum computing, many algorithms are known, eg

- Harrow-Hassidim-Lloyd algorithm for sparse linear systems
- Block-encoding method for general linear systems

A useful way to construct block-encoding of a matrix is based on the QRAM data structure

But assuming access to QRAM, there exist (quantum-inspired) classical algorithms that can also achieve poly-log dependence on the dimension [Ewin Tang 2018]

Background

Solving linear systems is a fundamental problem in machine learning

In the field of quantum computing, many algorithms are known, eg

- Harrow-Hassidim-Lloyd algorithm for sparse linear systems
- Block-encoding method for general linear systems

A useful way to construct block-encoding of a matrix is based on the QRAM data structure

But assuming access to QRAM, there exist (quantum-inspired) classical algorithms that can also achieve poly-log dependence on the dimension [Ewin Tang 2018]

As a result, exponential quantum speedups for many machine learning problems reduce to polynomial quantum speedups

Background

Solving linear systems is a fundamental problem in machine learning

In the field of quantum computing, many algorithms are known, eg

- Harrow-Hassidim-Lloyd algorithm for sparse linear systems
- Block-encoding method for general linear systems

A useful way to construct block-encoding of a matrix is based on the QRAM data structure

But assuming access to QRAM, there exist (quantum-inspired) classical algorithms that can also achieve poly-log dependence on the dimension [Ewin Tang 2018]

As a result, exponential quantum speedups for many machine learning problems reduce to polynomial quantum speedups

Question: How large is this polynomial speedup?

Main results

We focus on the solving of linear systems $A\mathbf{x} = \mathbf{b}$ with QRAM:

Algorithm	Complexity	Ref.	Assumptions
Quantum	$\widetilde{O}(\kappa_F)$	[1]	
Randomized	$\widetilde{O}(s\kappa_F^2)$	[2]	row sparse
classical	$\widetilde{O}(s\mathrm{Tr}(A)\|A^+\|)$	[3]	sparse, SPD
Quantum-inspired	$\widetilde{O}(\kappa_F^6\kappa^6/\epsilon^4)$	[4]	
classical	$\widetilde{O}(\kappa_F^6\kappa^2/\epsilon^2)$	Our	
	$\widetilde{O}(\mathrm{Tr}(A)^3\|A^+\|^3\kappa/\epsilon^2)$	Our	SPD

$\kappa_F = \|A\|_F\|A^{-1}\|$, [1] $\quad \kappa = \|A\|\|A^{-1}\|$, $\quad s =$ row sparsity,
SPD = symmetric positive definite.

[1] Chakraborty, Gilyén, Jeffery 2018 [2] Strohmer, Vershynin 2009
[3] Leventhal, Lewis 2010 [4] Gilyén, Song, Tang 2020

[1] $\|A\|_F^2 = \sum_{i,j} |A_{ij}|^2$

Implication 1: A is row sparse

Classical:
QRAM model $\Rightarrow \widetilde{O}(s\|A\|_F^2\|A^{-1}\|^2)$

Quantum:

- column sparse + sparse access input model
  $\Rightarrow \widetilde{O}(s\|A\|_{\max}\|A^{-1}\|)$ [2]
  $\Rightarrow$ large quantum speedup is possible
- column dense + QRAM model
  $\Rightarrow \widetilde{O}(\|A\|_F\|A^{-1}\|)$
  $\Rightarrow$ quadratic quantum speedup when $s = \widetilde{O}(1)$

Both achieve poly-log dependence on ϵ

[2] $\|A\|_{\max} = \max_{i,j} |A_{ij}|$

Implication 2: A is row sparse

The output:

- **Classical:** a sparse vector
- **Quantum:** a quantum state

If to estimate the norm of the solution (Assume QRAM + column dense):

- **Classical:** $\widetilde{O}(s\|A\|_F^2\|A^{-1}\|^2)$
- **Quantum:** $\widetilde{O}(\|A\|_F\|A^{-1}\|/\epsilon)$

$\Rightarrow$ Classical algorithm is better in terms of the dependence on ϵ

Implication 3: A is sparse and SPD

Classical:

- $\widetilde{O}(s\mathrm{Tr}(A)\|A^{-1}\|)$
- the output is a sparse vector

Quantum:

- $\widetilde{O}(s\|A\|_{\max}\|A^{-1}\|)$
- the output is a quantum state

$\Rightarrow$ May have no quantum speedup if $\mathrm{Tr}(A) = \widetilde{\Theta}(\|A\|_{\max})$.

Good news: For some SPD linear systems, quantum computers can have quadratically better dependence on the condition number [Davide, Dunjko, arXiv:2101.11868]

Implication 4: A is dense in the QRAM model

Classical: $\widetilde{O}(\|A\|_F^6\|A\|^2\|A^{-1}\|^8/\epsilon^2)$

This improves the previous best result $\widetilde{O}(\|A\|_F^6\|A\|^6\|A^{-1}\|^{12}/\epsilon^4)$ of Gilyén, Song, Tang 2020

The output is a classical analogue of the quantum output

Quantum: $\widetilde{O}(\|A\|_F\|A^{-1}\|)$

$\Rightarrow$ Large polynomial quantum speedup still exists.

The main technique: Kaczmarz method

Background:

- first discovered by the Polish mathematician Stefan Kaczmarz in 1937.

- rediscovered in the field of image reconstruction from projections by Richard Gordon, Robert Bender, and Gabor Herman in 1970.

The main technique: Kaczmarz method

It is an iterative algorithm for solving linear equation systems $A\mathbf{x} = \mathbf{b}$

Assume A is $m \times n$. Let $\mathbf{x}_0$ be an arbitrary initial approximation to the solution. For $k = 0, 1, \ldots$, compute

$$\mathbf{x}_{k+1} = \mathbf{x}_k + \frac{b_{r_k} - \langle A_{r_k *} | \mathbf{x}_k \rangle}{\|A_{r_k *}\|^2} A_{r_k *},$$

where $A_{r_k *}$ is the r_k-th row of A, and r_k is chosen from $\{1, ..., m\}$ randomly with probability proportional to $\|A_{r_k *}\|^2$

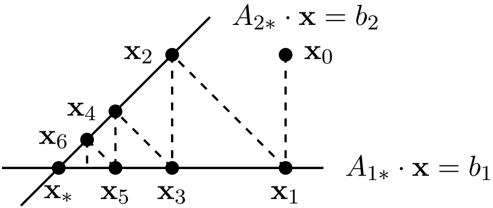

Figure 1: An illustration of Kaczmarz algorithm when $m = 2$.

Conclusions and outlooks

Conclusions:

- We reduced the gap between quantum and quantum-inspired classical algorithm for linear equations from $\kappa_F : \kappa_F^6\kappa^6/\epsilon^4$ to $\kappa_F : \kappa_F^6\kappa^2/\epsilon^2$.
- In the row sparse case, the quantum speedup is quadratic if assuming access to QRAM.

Outlooks:

- Reduce the dependence of quantum-inspired classical algorithm on $\|A\|_F$.

  Maybe generalized Kaczmarz methods,
  e.g., [arXiv:1712.09677, arXiv:1706.01108, arXiv:1902.09946]

Thank you!